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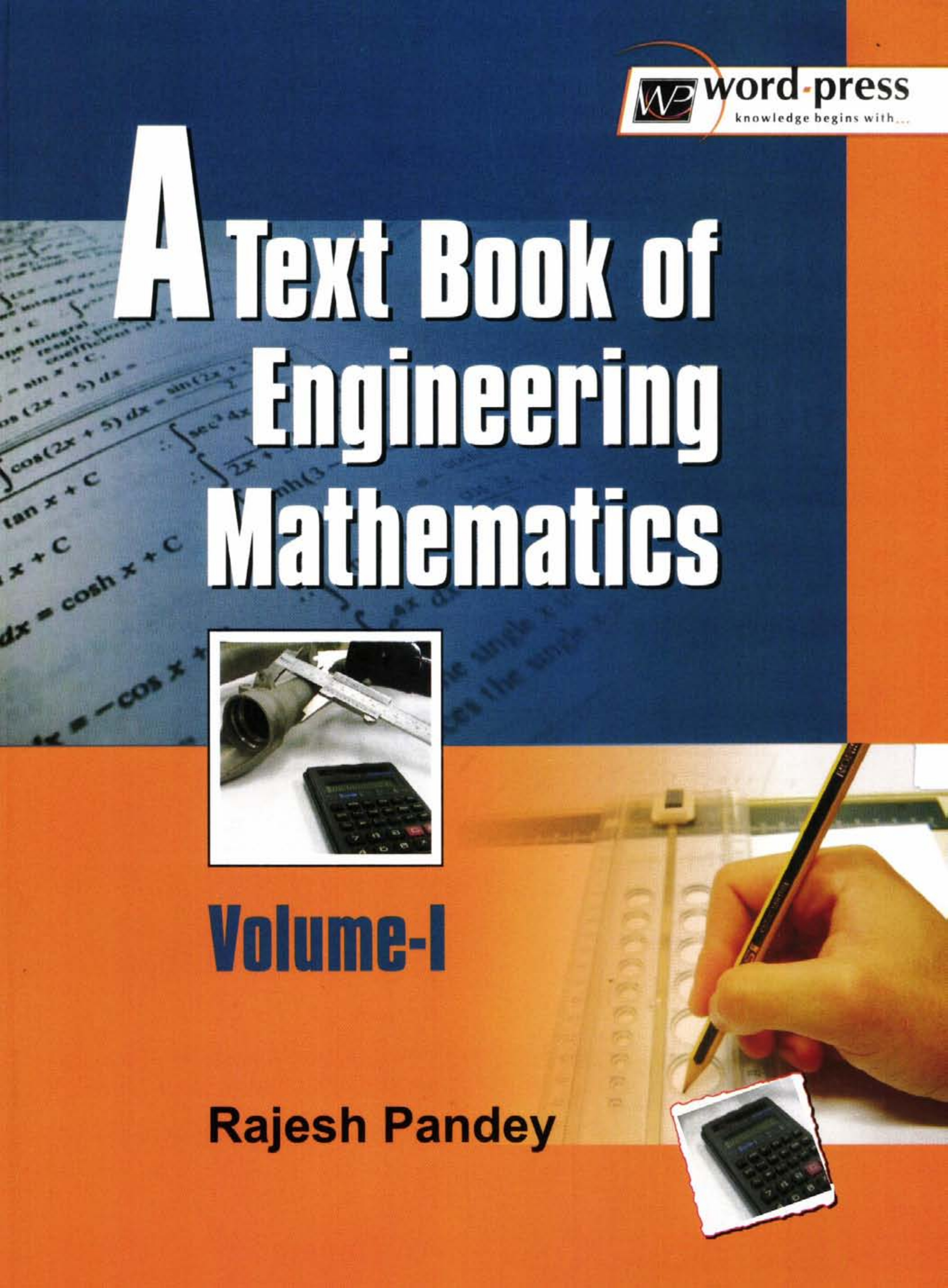
# A Text Book of Engineering Mathematics

## Volume-I



**Rajesh Pandey**

# A Text Book of ENGIEERING MATHEMATICS

## VOLUME - I

**Dr. Rajesh Pandey**
M.Sc., Ph.D.
Assistant Professor/Reader
Department of Mathematics
Sherwood College of Engineering,
Research and Technology Lucknow,
Faizabad Road, Barabanki (U.P.)

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# Basic Results and Concepts

## I. GENERAL INFORMATION

### 1. Greek Letters Used

| | | | |
|---|---|---|---|
| α alpha | θ theta | κ kappa | τ tau |
| β beta | ϕ phi | μ mu | χ chi |
| γ gamma | ψ psi | ν nu | ω omega |
| δ delta | ξ xi | π pi | Γ cap. gamma |
| ε epsilon | η eta | ρ rho | Δ cap. delta |
| i iota | ζ zeta | σ sigma | Σ cap. sigma |
| | λ lambda | | |

### 2. Some Notations

| | | |
|---|---|---|
| ∈ belongs to | ∪ union | ∉ doesnot belong to |
| ∩ intersection | ⇒ implies | / such that |
| ⇔ implies and implied by | | |

### 3. Unit Prefixes Used

| Multiples and Submultiples | Prefixes | Symbols |
|---|---|---|
| $10^3$ | kilo | k |
| $10^2$ | hecto | h |
| 10 | deca | da |
| $10^{-1}$ | deci* | d |
| $10^{-2}$ | centi* | c |
| $10^{-3}$ | milli | m |
| $10^{-6}$ | micro | μ |

* The prefixes 'deci' and 'centi' are only used with the metre, e.g., Centimeter is a recognized unit of length but Centigram is not a recognized unit of mass.

### 4. Useful Data

$e = 2.7183$ $1/e = 0.3679$ $\log_e 2 = 0.6931$ $\log_e 3 = 1.0986$

$\pi = 3.1416$ $1/\pi = 0.3183$ $\log_e 10 = 2.3026$ $\log_{10} e = 0.4343$

$\sqrt{2} = 1.4142$ $\sqrt{3} = 1.732$ 1 rad. = $57^0 17'45''$ $1^0 = 0.0174$ rad.

### 5. Systems of Units

| Quantily | F.P.S. System | C.G.S. System | M.K.S. System |
|---|---|---|---|
| Length | foot (ft) | centimetre (cm) | metre (m) |
| Mass | pound (lb) | gram (gm) | kilogram (kg) |
| Time | second (sec) | second (sec) | second (sec) |
| Force | lb. wt. | dyne | newton (nt) |

### 6. Conversion Factors

| | |
|---|---|
| 1 ft. = 30.48 cm = 0.3048 m | 1m = 100 cm = 3.2804 ft. |
| 1 $ft^2 = 0.0929\ m^2$ | 1 acre = 4840 $yd^2$ = 4046.77 $m^2$ |
| 1$ft^3 = 0.0283\ m^3$ | 1 $m^3$ = 35.32 $ft^3$ |
| 1 m/sec = 3.2804 ft/sec. | 1 mile /h = 1.609 km/h. |

## II. ALGEBRA

1. Quadratic Equation : $ax^2 + bx + c = 0$ has roots

$$\alpha = \frac{-b + \sqrt{(b^2 - 4ac)}}{2a},\ \beta = \frac{-b - \sqrt{(b^2 - 4ac)}}{2a}$$

$$\alpha + \beta = -\frac{b}{a},\quad \alpha\beta = \frac{c}{a}.$$

Roots are equal if $b^2 - 4ac = 0$

Roots are real and distinct if $b^2 - 4ac > 0$

Roots are imaginary if $b^2 - 4ac < 0$

**2. Progressions**

(i) Numbers a, a + d, a + 2d. ...... are said to be in Arithmetic Progression (A.P.)

Its nth term $T_n = a + \overline{n-1}\ d$ and sum $S_n = \frac{n}{2}(2a + \overline{n-1}\ d)$

(ii) Numbers a, ar, $ar^2$, ...... are said to be in Geometric Progression (G.P.)

Its nth term $T_n = ar^{n-1}$ and sum $S_n = \frac{a(1 - r^n)}{1 - r}$, $S_\infty = \frac{a}{1 - r}$ $(r < 1)$

(iii) Numbers 1/a, 1/(a + d), 1/(a + 2d),..... are said to be in Harmonic Progression (H.P.) (i.e., a sequence is said to be in H.P. if its reciprocals are in A.P. Its nth term $T_n = 1/(a + \overline{n-1}\ d)$. )

(iv) If a and b be two numbers then their

Arithmetic mean = $\frac{1}{2}(a + b)$, Geometric mean = $\sqrt{ab}$, Harmonic mean = 2ab/(a + b)

(v) Natural numbers are 1,2,3 ...,n.

$$\Sigma n = \frac{n(n+1)}{2},\ \Sigma n^2 = \frac{n(n+1)(2n+1)}{6},\ \Sigma n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

(vi) Stirling's approximation. When n is large $n! \sim \sqrt{2\pi n}\ .\ n^n\ e^{-n}$.

### 3. Permutations and Combinations

$$^{n}p_r = \frac{n!}{(n-r)!};\ ^{n}C_r = \frac{n!}{r!\,(n-r)!} = \frac{^{n}P_r}{r!}$$

$$n_{C_{n-r}} = n_{C_r},\ n_{C_0} = 1 = n_{C_n}$$

### 4. Binomial Theorem

(i) When n is a positive integer

$(1 + x)^n = 1 + {}^nC_1\,x + {}^nC_2\,x^2 + {}^nC_3\,x^3 + \ldots\ldots. + {}^nC_n\,x^n$.

(ii) When n is a negative integer or a fraction

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)\,(n-2)}{1.2.3}x^3 + \ldots..\infty.$$

### 5. Indices

(i) $a^m\ .\ a^n = a^{m+n}$

(ii) $(a^m)^n = a^{mn}$

(iii) $a^{-n} = 1/a^n$

(iv) n $\sqrt{a}$ (i.e., nth root of a) = $a^{1/n}$.

### 6. Logarithms

(i) Natural logarithm log x has base e and is inverse of $e^x$.

Common logarithm $\log_{10}x$ = M log x where M = $\log_{10}e$ = 0.4343.

(ii) $\log_a 1 = 0$; $\log_a 0 = -\infty (a > 1)$; $\log_a a = 1$.

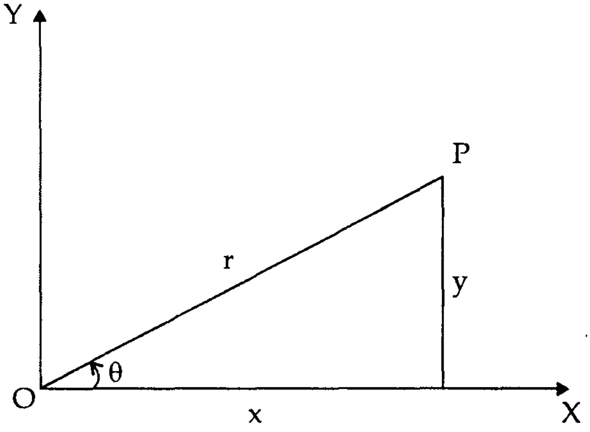
(iii) log (mn) = log m + logn ; log (m/n) = log m - log n; log ($m^n$) = n log m.

## III. GEOMETRY

1. Coordinates of a point : Cartesian (x ,y) and polar (r , θ).

Then $x = r\cos\theta$, $y = r\sin\theta$

or $r = \sqrt{(x^2 + y^2)}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.



Distance between two points

$(x_1, y_1)$ and $(x_2, y_2) = \sqrt{\left[(x_2 - x_1)^2 + (y_2 - y_1)^2\right]}$

Points of division of the line joining $(x_1, y_1)$ and $(x_2, y_2)$ in the ration $m_1 : m_2$ is

$$\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$$

In a triangle having vertices $(x_1, y_1)$, $(x_2, y_2)$ and $(x_3, y_3)$

(i) area $= \dfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

(ii) Centroid (point of intersection of medians) is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

(iii) Incentre (point of intersection of the internal bisectors of the angles) is

$$\left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c}\right)$$

where a, b, c are the lengths of the sides of the triangle.

(iv) Circumcentre is the point of intersection of the right bisectors of the sides of the triangle.

(v) Orthocentre is the point of intersection of the perpendiculars drawn from the vertices to the opposite sides of the triangle.

**2. Straight Line**

(i) Slope of the line joining the points $(x_1, y_1)$ and $(x_2, y_2) = \dfrac{y_2 - y_1}{x_2 - x_1}$

Slope of the line $ax + by + c = 0$ is $-\dfrac{a}{b}$ i.e., $-\dfrac{\text{coeff,of } x}{\text{coeff,of } y}$

(ii) Equation of a line:

(a) having slope m and cutting an intercept c on y - axis is $y = mx + c$.

(b) cutting intercepts a and b from the axes is $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

(c) passing through $(x_1, y_1)$ and having slope m is $y - y_1 = m(x - x_1)$

(d) Passing through $(x_1, y_2)$ and making an $\angle\theta$ with the x - axis is

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

(e) through the point of intersection of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is $a_1x + b_1y + c_1 + k(a_2x + b_2y + c_2) = 0$

(iii) Angle between two lines having slopes $m_1$ and $m_2$ is $\tan^{-1} \dfrac{m_1 - m_2}{1 - m_1m_2}$

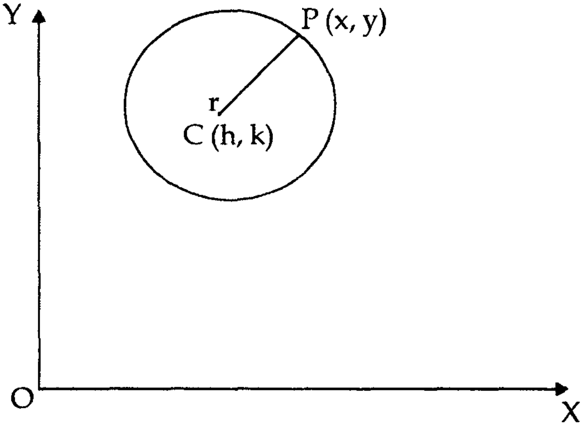
Two lines are parallel if $m_1 = m_2$

Two lines are perpendicular if $m_1m_2 = -1$

Any line parallel to the line $ax + by + c = 0$ is $ax + by + k = 0$

Any line perpendicular to $ax + by + c = 0$ is $bx - ay + k = 0$

(iv) Length of the perpendicular from $(x_1, y_1)$of the line $ax + by + c = 0$. is $\frac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}}$.



**3. Circle**

(i) Equation of the circle having centre (h, k) and radius r is
$(x - h)^2 + (y - k)^2 = r^2$

(ii) Equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle having centre (-g, -f) and radius = $\sqrt{(g^2 + f^2 - c)}$.

(iii) Equation of the tangent at the point $(x_1, y_1)$ to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

(iv) Condition for the line $y = mx + c$ to touch the circle
$x^2 + y^2 = a^2$ is $c = a\sqrt{(1 + m^2)}$.

(v) Length of the tangent from the point $(x_1, y_1)$ to the circle
$x^2 + y^2 + 2gx + 2fy + c = 0$ is $\sqrt{(x_1^2 - y_1^2 + 2gx_1 + 2fy_1 + c)}$.

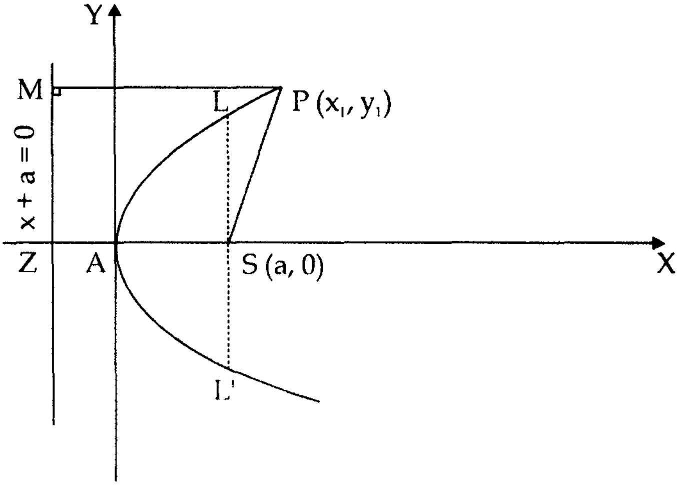
**4. Parabola**

(i) Standard equation of the parabola is $y^2 = 4ax$.

Its parametric equations are $x = at^2$, $y = 2at$.

Latus - rectum LL' = 4a, Focus is S (a,0)

Directrix ZM is $x + a = 0$.



(ii) Focal distance of any point P $(x_1, y_1)$ on the parabola
$y^2 = 4ax$ is $SP = x_1 + a$

(iii) Equation of the tangent at $(x_1, y_1)$ to the parabola
$y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$

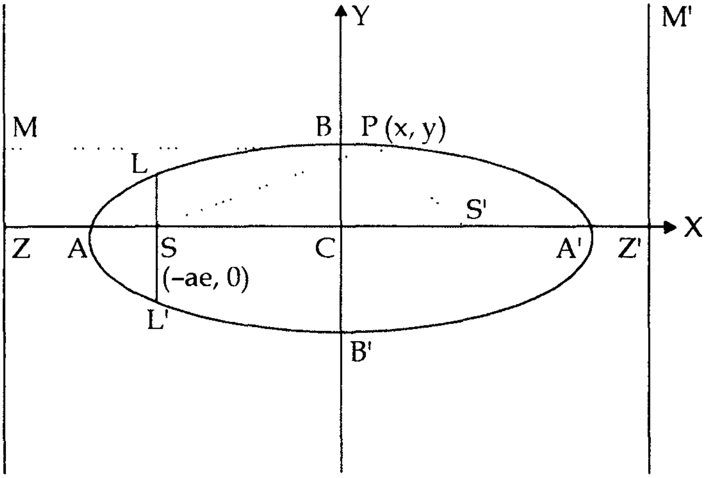
(iv) Condition for the line $y = mx + c$ to touch the parabola
$y^2 = 4ax$ is $c = a/m$.

(v) Equation of the normal to the parabola $y^2 = 4ax$ in terms of its slope m is
$y = mx - 2am - am^3$.

**5. Ellipse**

(i) Standard equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$



Its parametric equations are

$x = a\cos\theta,$ $\qquad y = b\sin\theta.$

Eccentricity $e = \sqrt{(1 - b^2/a^2)}$.

Latus - rectum LSL' = $2b^2/a$.

Foci S $(-ae, 0)$ and S' $(ae, 0)$

Directrices ZM $(x = -a/e)$ and Z'M' $(x = a/e.)$

(ii) Sum of the focal distances of any point on the ellipse is equal to the major axis i.e.,

SP + S'P = 2a.

(iii) Equation of the tangent at the point $(x_1, y_1)$ to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

(iv) Condition for the line $y = mx + c$ to touch the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } c = \sqrt{(a^2m^2 + b^2)}.$$

**6. Hyperbola**

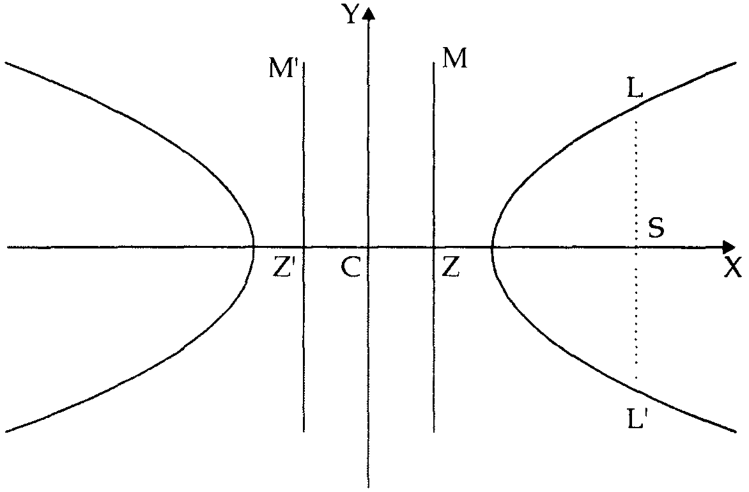
(i) Standard equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Its parametric equations are

$x = a\sec\theta,\ y = b\tan\theta.$

Eccentricity $e = \sqrt{(1 + b^2/a^2)},$



Latus - rectum LSL' = $2b^2/a$.

Directrices ZM $(x = a/e)$ and Z'M' $(x = -a/e)$.

(ii) Equation of the tangent at the point $(x_1, y_1)$ to the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

(iii) Condition for the line $y = mx + c$ to touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $c = \sqrt{(a^2m^2 - b^2)}$

(iv) Asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $\frac{x}{a} + \frac{y}{b} = 0$ and $\frac{x}{a} - \frac{y}{b}\ 0$.

(v) Equation of the rectangular hyperbola with asymptotes as axes is $xy = c^2$. Its parametric equations are $x = ct$, $y = c/t$.

**7. Nature of the a Conic**

The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents

(i) a pair of lines, if $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ $(= \Delta) = 0$

(ii) a circle, if $a = b$, $h = 0$, $\Delta \neq 0$

(iii) a parabola, if $ab - h^2 = 0$,c $\Delta \neq 0$

(iv) an ellipse, if $ab - h^2 > 0$, $\Delta \neq 0$

(v) a hyperbola, if $ab - h^2 > 0$, $\Delta \neq 0$

and a rectangular hyperbola if in addition, $a + b = 0$.

**8. Volumes and Surface Areas**

| **Solid** | **Volume** | **Curved Surface Area** | **Total Surface Area** |
|---|---|---|---|
| Cube (side a) | $a^3$ | $4a^2$ | $6a^2$ |
| Cuboid (length l, breadth b, height h) | $lbh$ | $2(l + b)h$ | $2(lb + bh + hl)$ |
| Sphere (radius r) | $\frac{4}{3}\pi r^3$ | – | $4\pi r^2$ |
| Cylinder (base radius r, height h) | $\pi r^2 h$ | $2\pi rh$ | $2\pi r(r + h)$ |
| Cone | $\frac{1}{3}\pi r^2 h$ | $\pi rl$ | $\pi r(r + l)$ |
| where slant height $l$ is given by $l = \sqrt{(r^2 + h^2)}$. | | | |

## IV. TRIGONOMETRY

1.

| $\theta^o = 0$ | 0 | 30 | 45 | 60 | 90 | 180 | 270 | 360 |
|---|---|---|---|---|---|---|---|---|
| $\sin\theta$ | 0 | ½ | $1/\sqrt{2}$ | $\frac{\sqrt{3}}{2}$ | 1 | 0 | $-1$ | 0 |
| $\cos\theta$ | 1 | $\frac{\sqrt{3}}{2}$ | $1/\sqrt{2}$ | 1/2 | 0 | $-1$ | 0 | 1 |
| $\tan\theta$ | 0 | $1/\sqrt{3}$ | 1 | $\sqrt{3}$ | $\infty$ | 0 | $-\infty$ | 0 |

2. Any t-ratio of $(n.\ 90^0 \pm \theta)$ = ± same ratio of θ, when n is even.

= ± co - ratio of θ, when n is odd.

The sign + or — is to be decided from the quadrant in which $n.\ 90^0 \pm \theta$ lies.

e.g., $\sin 570^0 = \sin(6 \times 90^0 + 30^0) = -\sin 30^0 = -\frac{1}{2}$;

$\tan 315^0 = \tan(3 \times 90^0 + 45^0) = -- \cot 45^0 = -1$.

3. $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \pm \sin A \sin B$

$\sin 2A = \cos^2 A \cos A = 2 \tan A/(1 + \tan^2 A)$

$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1 = \frac{1 - \tan^2 A}{1 + \tan^2 A}$.

4. $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 + \tan A \tan B}$; $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$.

5. $\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$

$\cos A \sin B = \frac{1}{2}[\sin(A + B) - \sin(A - B)]$

$\mathrm{coa}\ A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$

$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$.

6. $\sin C + \sin D = 2 \sin\frac{C + D}{2} \cos\frac{C - D}{2}$

$\sin C - \sin D = 2 \cos\frac{C + D}{2} \sin\frac{C - D}{2}$

$\cos C + \cos D = 2 \cos\frac{C + D}{2} \cos\frac{C - D}{2}$

$\cos C - \cos D = -2 \sin\frac{C + D}{2} \sin\frac{C - D}{2}$

7. $a \sin x + b \cos x = r \sin(x + \theta)$

$a \cos x + b \sin x = r \cos(x - \theta)$

where a = r cos, θ, b = r sinθ so that r = $\sqrt{(a^2 + b^2)}$, θ $\tan^{-1}$ (b/a)

8. In any ΔABC:

(i) a/sin A = b/sin B = c/sin C (sin formula)

(ii) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$. (cosine formula)

(iii) a = b cos C + c cos B (Projection formula)

(iv) Area of ΔABC = $\frac{1}{2}bc \sin A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{1}{2}(a + b + c)$.

**9. Series**

(i) Exponential Series: $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots\ldots\infty$

(ii) sin x, cos x, sin hx, cos hx series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\infty,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots\infty$$

$$\sin h\, x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots\infty, \quad \cos h\, x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots\infty$$

(iii) Log series

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots\infty, \quad \log(1 - x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\infty\right)$$

(iv) Gregory series

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots\infty, \tan h^{-1} x = \frac{1}{2} \log \frac{1 + x}{1 - x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\infty.$$

10. (i) Complex number : $z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

(ii) Euler's theorem: $\cos\theta + i\sin\theta = e^{i\theta}$

(iii) Demoivre's theorem: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

11. (i) Hyperbolic functions: $\sin h\, x = \dfrac{e^x - e^{-x}}{2}$; $\cos h\, x = \dfrac{e^x + e^{-x}}{2}$;

$$\tan h\, x = \frac{\sin h\, x}{\cosh x};\ \cot h\, x = \frac{\cos h\, x}{\sin h\, x};\ \sec h\, x = \frac{1}{\cos h\, x};\ \operatorname{cosec} h\, x = \frac{1}{\sin hx}$$

(ii) Relations between hyperbolic and trigonometric functions:

sin ix = i sin h x ; cos h x = cos h x ; tan ix = i tan h x.

(iii) Inverse hyperbolic functions;

$$\sin h^{-1}x = \log[x + \sqrt{x^2 + 1}];\ \cosh^{-1}x = \log[x + \sqrt{x^2 - 1}];\ \tan h^{-1} x = \frac{1}{2} \log \frac{1+x}{1 - x}.$$

## V. CALCULUS

### 1. Standard limits:

(i) $\underset{x \to a}{\mathrm{Lt}} \dfrac{x^n - a^n}{x - a} = na^{n-1}$, n any rational number

(ii) $\underset{x \to 0}{\mathrm{Lt}} \dfrac{\sin x}{x} = 1$

(iii) $\underset{x \to 0}{\mathrm{Lt}} (1 + x)^{1/x} = e$

(iv) $\underset{x \to \infty}{\mathrm{Lt}} x^{1/x} = 1$

(v) $\underset{x \to 0}{\mathrm{Lt}} \dfrac{a^x - 1}{x} = \log_e a.$

### 2. Differentiation

(i) $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ $\qquad$ $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\,du/dx - u\,dv/dx}{v^2}$

$\dfrac{du}{dx} = \dfrac{du}{dy} \cdot \dfrac{dy}{dx}$ (chain Rule) $\qquad$ $\dfrac{d}{dx}(ax + b)^n = n(ax + b)^{n-1} \cdot a$

(ii) $\dfrac{d}{dx}(e^x) = e^x$ $\qquad$ $\dfrac{d}{dx}(a^x) = a^x \log_e a$

$\dfrac{d}{dx}(\log_e x) = 1/x$ $\qquad$ $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \log a}.$

(iii) $\dfrac{d}{dx}(\sin x) = \cos x$ $\qquad$ $\dfrac{d}{dx}(\cos x) = -\sin x$

$\dfrac{d}{dx}(\tan x) = \sec^2 x$ $\qquad$ $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$\dfrac{d}{dx}(\sec x) = \sec x \tan x$ $\qquad$ $\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x.$

(iv) $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{(1 - x^2)}}$ $\qquad$ $\dfrac{d}{dx}(\cos^{-1} x) = \dfrac{-1}{\sqrt{(1 - x^2)}}$

$\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1 + x^2}$ $\qquad$ $\dfrac{d}{dx}(\cot^{-1} x) = \dfrac{-1}{1 + x^2}$

$\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{x\sqrt{(x^2 - 1)}}$ $\qquad$ $\dfrac{d}{dx}(\operatorname{cosec}^{-1} x) = \dfrac{-1}{x\sqrt{(x^2 - 1)}}.$

(v) $\dfrac{d}{dx}(\sinh x) = \cosh x$ $\qquad$ $\dfrac{d}{dx}(\cosh x) = \sinh x$

$\dfrac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$ $\qquad$ $\dfrac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x.$

(vi) $D^n (ax + b)^m = m(m - 1)(m - 2) \ldots\ldots (m - n + 1)(ax + b)^{m-n} \cdot a^n$

$D^n \log(ax + b) = (-1)^{n-1}(n - 1)!\, a^n/(ax + b)^n$

$D^n (e^{mx}) = m^n e$ $\qquad$ $D^n (a^{mx}) = m^n (\log a)^n \cdot a^{mx}$

$$D^n \begin{bmatrix} \sin(ax+b) \\ \cos(bx+c) \end{bmatrix} = (a^2+b^2)^{n/2}\, e^{ax} \begin{bmatrix} \sin(bx+c+n\tan^{-1} b/a) \\ \cos(bx+c+n\tan^{-1} b/a) \end{bmatrix}.$$

(vii) Leibnitz theorem: $(uv)_n$
$= u_n + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \ldots\ldots + {}^nC_r u_{n-r} v_r + \ldots\ldots + {}^nC_n v_n.$

**3. Integration**

(i) $\int x^n \, dx = \dfrac{x^{n+1}}{n+1}$ $(n \neq -1)$ $\qquad \int \dfrac{1}{x} \, dx = \log_e x$

$\int e^x \, dx = e^x \qquad \int a^x \, dx = a^x / \log_e a$

(ii) $\int \sin x \, dx = -\cos x \qquad \int \cos x \, dx = \sin x$

$\int \tan x \, dx = -\log \cos x \qquad \int \cot x \, dx = \log \sin x$

$$\int \sec x \, dx = \log(\sec x + \tan x) = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

$$\int \operatorname{cosec} x \, dx = \log(\operatorname{cosec} x - \cot x) = \log \tan\left(\frac{x}{2}\right)$$

$\int \sec^2 x \, dx = \tan x \qquad \int \operatorname{cosec}^2 x \, dx = -\cot x.$

(iii) $\int \dfrac{dx}{a^2+x^2} = \dfrac{1}{a} \tan^{-1} \dfrac{x}{a} \qquad \int \dfrac{dx}{\sqrt{(a^2-x^2)}} = \sin^{-1} \dfrac{x}{a}$

$\int \dfrac{dx}{a^2-x^2} = \dfrac{1}{2a} \log \dfrac{a+x}{a-x} \qquad \int \dfrac{dx}{\sqrt{(a^2+x^2)}} = \sin h^{-1} \dfrac{x}{a}$

$\int \dfrac{dx}{x^2-a^2} = \dfrac{1}{2a} \log \dfrac{x-a}{a+x} \qquad \int \dfrac{dx}{\sqrt{(x^2-a^2)}} = \cos h^{-1} \dfrac{x}{a}.$

(iv) $\int \sqrt{(a^2-x^2)} \, dx = \dfrac{x\sqrt{(a^2-x^2)}}{2} + \dfrac{a^2}{2} \sin^{-1} \dfrac{x}{a}$

$$\int \sqrt{(a^2+x^2)} \, dx = \frac{x\sqrt{(a^2+x^2)}}{2} + \frac{a^2}{2} \sinh^{-1} \frac{x}{a}. = \frac{x}{2}\sqrt{(a^2+x^2)} + \frac{a^2}{2} \log \frac{x+\sqrt{(a^2+x^2)}}{a}$$

$$\int \sqrt{(x^2-a^2)} \, dx = \frac{x\sqrt{(x^2-a^2)}}{2} + \frac{a^2}{2} \cosh^{-1} \frac{x}{a}. = \frac{x}{2}\sqrt{(x^2-a^2)} - \frac{a^2}{2} \log \frac{x+\sqrt{(x^2-a^2)}}{a}$$

(v) $\int e^{ax} \sin bx \, dx = \dfrac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$$

(vi) $\int \sin h\ x\,dx = \cos h\ x$ $\qquad$ $\int \cos h\ x\,dx = \sin h\ x$

$\int \tan h\ x\,dx = \log \cos h\ x$ $\qquad$ $\int \cot h\ x\,dx = \log \sin h\ x$

$\int \sec h^2\ x\,dx = \tan h\ x$ $\qquad$ $\int \operatorname{cosech}^2\ x\,dx = -\ \cot h\ x.$

(vii) $\int_0^{\pi/2} \sin^n\ x\,dx = \int_0^{\pi/2} \cos^n\ x\,dx$

$$= \frac{(n-1)(n-3)(n-5)\ldots.}{n(n-2)(n-4)} \times \left(\frac{\pi}{2}, \text{ only if n is even}\right)$$

$$\int_0^{\pi/2} \sin^m\ x \cos^n\ x\,dx = \frac{(m-1)(m-3)\ldots. \times (n-1)(n-3)\ldots.}{(m+n)(m+n-2)(m+n-4)\ldots.}$$

$$\times \left(\frac{\pi}{2}, \text{ only if both m and n are even}\right)$$

(viii) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$\int_{-a}^{a} f(x)\,dx = 2 \int_0^a f(x)\,dx$, if f (x) is an even function.

= 0, if f(x) is an odd function.

$\int_0^{2a} f(x)\,dx = 2 \int_0^a f(x)\,dx$, if $f(2a - x) = f(x)$

= 0, if $f(2a - x) = -f(x)$.

## VI. VECTIORS

1. (i) If $R = x\hat{i} + y\hat{j} + z\hat{k}$ then $r = |R| = \sqrt{(x^2 + y^2 + z^2)}$

(ii) $\overline{PQ}$ = position vector of Q-position vector of P.

2. If $A = a_1 I + a_2 J + a_3 K$, $B = b_1 I + b_2 J + b_3 K$, then

(i) Scalar product: $A \,.\, B = ab\cos\theta = a_1b_1 + a_2b_2 + a_3b_3$

(ii) Vector product: $A \times B = ab \sin\theta\ N$ = Area of the parallelogram having A + B as sides

$$= \begin{vmatrix} I & J & K \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(iii) BA ⊥ IF $A \,.\, B = 0$ and A is parallel to B if $A \times B = 0$

3. (i) Scalar triple product $[A\ B\ C] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ = Volume of parallelopiped

(ii) If $[A\ B\ C] = 0$, then A, B, C are coplanar

(iii) Vector triple product $A \times (B \times C) = (A \,.\, C)\,B - (A \,.\, B)\,C$

$(A \times B) \times C = (C \,.\, A)\,B - (C \,.\, B)\,A$

4. (i) grad $f = \nabla f = \frac{\partial f}{\partial x} I + \frac{\partial f}{\partial y} J + \frac{\partial f}{\partial z} K$

div $F = \nabla F = \frac{\partial f_1}{dx} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

curl $F = \nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$ where $F = f_1 I + f_2 J + f_3 K$

(ii) If div F = 0, then F is called a solenoidal vector

(iii) If curl F = 0 then F is called an irrotational vector

5. Velocity = $dR/dt$; Acceleration = $d^2R/dt^2$;

Tangent vector = $dR/dt$ ; Normal vector = $\nabla \phi$

6. Green's theorem : $\int_C (\phi dx + \psi dy) = \int\int C\left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dxdy$

Stoke's theorem: $\int_C F \,.\, dR = \int_S \text{curl } F \,.\, N \, ds$

Gauss divergence theorem: $\int_S F \,.\, N \, ds = \int_V \text{div } F \, dv$

**7. Coordinate systems**

| | Polar coordinates $(r, \theta)$ | Cylindrical coordinates $(\rho, \phi, z)$ | Spherical polar coordinates $(r, \theta, \phi)$ |
|---|---|---|---|
| Coordinate transformations | $x = r \cos\theta$<br>$y = r \sin\theta$ | $x = \rho \cos\phi$<br>$y = \rho \sin\phi$<br>$z = z$ | $x = r \sin\theta \cos\phi$<br>$y = r \sin\theta \sin\phi$<br>$z = r \cos\theta$ |
| Jacobian | $\frac{\partial(x, y)}{\partial(r, \theta)} = r$ | $\frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho$ | $\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin\theta$ |
| (Arc - length)$^2$ | $(ds)^2 = (dr)^2 + r^2 (d\theta)^2$<br>$dx\,dy = r d\theta\, dr$ | $(ds)^2 = (d\rho)^2 + \rho^2 (d\phi)^2 + (dz)^2$ | $(ds)^2 = (dr)^2 + r^2 (d\theta)^2 + (r \sin\theta)^2 (d\phi)^2$ |
| Volume- element | | $dV = \rho\, d\rho\, d\phi\, dz$ | $dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$ |

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# UNIT - 1

# Differential Calculus-I

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# Chapter 1

# Succesive Differentiation and Leibnitz's Theorm

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**Successive Differentiation**

**Definition and Notation :-** If y be a function of x, its differential coefficient dy/dx will be in general a function of x which can be differentiated. The differential coefficient of dy/dx is called the second differential coefficient of y. Similarly, the differential coefficient of the second differential coefficient is called the third differential coefficient, and so on. The successive differential coefficients of y are denoted by

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3} \ldots\ldots\ldots\ldots\ldots\ldots$$

the nth differential coefficient of y being $\frac{d^ny}{dx^n}$

Alternative methods of writing the nth differential coefficient are

$$\left(\frac{d}{dx}\right)^n y\,, D^ny, y_n, \frac{d^ny}{dx^n}, y^{(n)}$$

The Process to find the differential coefficient of a function again and again is called successive Differentiation.

Thus, if y = f(x), the successive differential coefficients of f(x) are

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots\ldots\ldots\ldots\ldots\ldots \frac{d^ny}{dx^n}$$

These are also denoted by :

(i) $y_1, y_2, y_3 \ldots\ldots\ldots\ldots y_n$

(ii) $y', y'', y''' \ldots\ldots\ldots\ldots y^n$

(iii) $Dy, D^2y, D^3y \ldots\ldots\ldots\ldots D^ny$

(iv) $f'(x), f''(x), f'''(x), \ldots\ldots\ldots\ldots f^n(x)$

**$n^{th}$ Derivatives of some standard functions :-**

**(1) $n^{th}$ derivative of $(ax+b)^m$ :-**

Let $y = (ax+b)^m$

Differentiating it w.r.t. x in succession, we get

$y_1 = m(ax+b)^{m-1}.a$

$y_2 = m(m-1)(ax+b)^{m-2} a^2$

$y_3 = m(m-1) (m-2) (ax+b)^{m-3} a^3$

Similarly, we can write

$y_n = m(m-1)\,(m-2)\,(ax+b)^{m-n}\,a^n$

$D^n\,(ax+b)^m = m(m-1)\,(m-2)\ldots\ldots\ldots(m-n+1)\,(ax+b)^{m-n}.\,a^n$

If m is a positive integer then R.H.S of the above can be written as

$$\frac{\underline{|m}}{\underline{|(m-n)}}\,(ax+b)^{m-n}.\,a^n$$

$$\text{Hence } D^n(ax+b)^m = \frac{\underline{|m}}{\underline{|(m-n)}}\,.a^n\,(ax+b)^{m-n}$$

**Deductions :**

(a) If $m = n$, then $D^n(ax+b)^n = \underline{|n}\,.a^n$

In particular, $D^n\,x^n = \underline{|n}$

$$D^{n-2}\,x^n = \frac{\underline{|n}}{\underline{|2}}\,x^2$$

$$D^{n-3}\,x^n = \frac{\underline{|n}}{\underline{|3}}\,x^3 \text{ etc.}$$

(b) If $m = -p$, where p is a positive integer, then

$$D^n(ax+b)^{-p} = (-1)^n\,\frac{\underline{|(p+n-1)}}{\underline{|(p-1)}}\,a^n\,(ax+b)^{-p-n}$$

(c) $D^n\,(ax+b)^{-1} = \dfrac{(-1)^n\,\underline{|n}}{(ax+b)^{n+1}}\,a^n$

(d) If $m < n$, then $D^n\,(ax+b)^m = 0$

**2. nth Derivative of $e^{ax}$ :**

Let $y = e^{ax}$

Differentiating w.r.t. x in succession we get

$y_1 = a\,e^{ax},\ \ y_2 = a^2\,e^{ax},\ y_3 = a^3\,e^{ax},$

Similarly we can write $y_n = a^n\,e^{ax}$

$\therefore D^n\,e^{ax} = a^n\,e^{ax}$

**3. nth derivative of $a^{mx}$ :-**

Let $y = a^{mx}$

Differentiation w.r.t x in succession, we get

$y_1 = m a^{mx}\log a$

$y_2 = m^2\,a^{mx}\,(\log a)^2$

Similarly, we can write $y_n = m^n\,a^{mx}\,(\log a)^n$

$\therefore\ D^n\,a^{mx} = m^n\,a^{mx}\,(\log a)^n$

In particular,

$D^n\,a^x = a^x\,(\log a)^n$

**4. nth Derivative of log (ax+b)**

Let $y = \log\,(ax+b)$

$y_1 = \frac{a}{ax+b} = a(ax+b)^{-1}$

$y_2 = (-1)\,(ax+b)^{-2}\,a^2$

$y_3 = (-1)\,(-2)\,(ax+b)^{-2}\,a^3$

Similarly we can write

$y_n = (-1)\,(-2)\ldots\ldots\ldots(-n+1)\,(ax+b)^{-n}\,a^n$

$= (-1)^{n-1}\,1.2.3\ldots\ldots\ldots\ldots\ldots\ldots(n-1)\,(ax+b)^{-n}\,a^n$

$= (-1)^{n-1}\,\underline{|(n-1)}\,.\,(ax+b)^{-n}.\,a^n$

$= \frac{(-1)^{n-1}a^n\underline{|n-1}}{(ax+b)^n}$

$\therefore\ D^n \log(ax+b) = \frac{(-1)^{n-1}a^n\underline{|(n-1)}}{(ax+b)^n}$

**5. n^th Derivative of sin (ax+b):-**

Let $y = \sin(ax+b)$

$\therefore\ y_1 = a\cos(ax+b)$

$y_1 = a\sin(ax+b+\frac{\pi}{2})$

$y_2 = a^2\cos(ax+b+\frac{\pi}{2})$

$y_2 = a^2\sin(ax+b+\frac{2\pi}{2})$

$y_3 = a^3\sin(ax+b+\frac{3\pi}{2})$

Similarly, we can write

$y_n = a^n\sin(ax+b+\frac{n\pi}{2})$

$\therefore\ D^n\sin(ax+b) = a^n\sin(ax+b+\frac{n\pi}{2})$

In particular,

$D^n\sin x = \sin(x+\frac{n\pi}{2})$

**6. n^th Derivative of cos (ax+b) :**

Let $y = \cos(ax+b)$

$\therefore\ y_1 = -a\sin(ax+b)$

$= a\cos(ax+b+\frac{\pi}{2})$

$y_2 = -a^2\sin(ax+b+\frac{\pi}{2})$

$= a^2 \cos (ax+b+2.\frac{\pi}{2})$

$y_3 = -a^3 \sin (ax+b+\frac{2\pi}{2})$

$= a^3 \cos (ax+b+\frac{3\pi}{2})$

Similarly, we can write

$y_n = a^n \cos (ax+b+\frac{n\pi}{2})$

$\therefore D^n \cos (ax+b) = a^n \cos (ax+b+\frac{n\pi}{2})$

In particular,

$\therefore D^n \cos x = \cos (x+\frac{n\pi}{2})$

**7. nth Derivative of $e^{ax} \sin (bx+c)$**

Let $y = e^{ax} \sin (bx+c)$

$y_1 = a\, e^{ax} \sin (bx+c)+ b\, e^{ax} \cos (bx+c)$

put $a = r \cos\phi$ and $b = r \sin\phi$

$\therefore r^2 = a^2+b^2$ and $\phi = \tan^{-1} (b/a)$

$y_1 = r\, e^{ax} [\cos\phi \sin (bx+c) +\sin\phi \cos (bx+c)]$

Similarly, we can obtain

$y_2 = r^2 e^{ax} \sin (bx+c+2\phi)$

$y_3 = r^3 e^{ax} \sin (bx+c+3\phi)$

Continuingthis process n times, we get

$y_n = r^n e^{ax} \sin (bx+c+n\phi)$

$\therefore D^n \{e^{ax} \sin (bx+c)\} = r^n. e^{ax} \sin(bx+c+n\phi)$

where $r = (a^2+b^2)^{1/2}$ and $\phi = \tan^{-1}\left(\frac{b}{a}\right)$

**8. nth Derivative of $e^{ax} \cos (bx+c)$ :**

Proceeding exactly as above, we get

$D^n \{e^{ax} \cos(bx+c)\}= r^n. e^{ax} \cos (bx+c+n\phi)$

where r and $\phi$ have the same meaning as above.

**Example 1:** if $y = \frac{1}{1-5x+6x^2}$, find $y_n$

(U.P.T.U. 2005)

**Solution :**

$y = \frac{1}{1-5x+6x^2}$

$$= \frac{1}{(2x-1)(3x-1)}$$

$$= \frac{2}{2x-1} - \frac{3}{3x-1}$$

$$\therefore\ y_n = 2D^n\left(\frac{1}{2x-1}\right) - 3D^n\left(\frac{1}{3x-1}\right)$$

$$= 2\left[\frac{(-1)^n \underline{|n}(2)^n}{(2x-1)^{n+1}}\right] - 3\left[\frac{(-1)^n \underline{|n}(3)^n}{(3x-1)^{n+1}}\right]$$

$$= (-1)^n \underline{|n}\left[\frac{(2)^{n+1}}{(2x-1)^{n+1}} - \frac{(3)^{n+1}}{(3x-1)^{n+1}}\right] \text{ Ans.}$$

**Example 2 :** if $y = \dfrac{x^2}{(x-1)^2(x+2)}$, find nth derivative of y.

(U.P.T.U. 2002)

**Solution :-** To split y into partial fractions
Let x-1 = z, then

$$y = \frac{1}{z^2} \cdot \frac{1+2z+z^2}{3+z}$$

$$= \frac{1}{z^2}\left(\frac{1}{3} + \frac{5z}{9} + \frac{4}{9}\frac{z^2}{3+z}\right)$$

$$= \frac{1}{3z^2} + \frac{5}{9z} + \frac{4}{9(3+z)}$$

$$= \frac{1}{3(x-1)^2} + \frac{5}{9(x-1)} + \frac{4}{9(x+2)}$$

Hence

$$y_n = \frac{(-1)^n \underline{|(n+1)}}{3(x-1)^{n+2}} + \frac{5(-1)^n \underline{|n}}{9(x-1)^{n+1}} + \frac{4(-1)^n \underline{|n}}{9(x+2)^{n+1}} \text{ Answer.}$$

**Example 3 :** Find the nth derivative of $\tan^{-1}\left(\dfrac{2x}{(1-x^2)}\right)$

(U.P.T.U. 2001)

**Solution :-** we have

$$\tan^{-1}\left(\frac{2x}{(1-x^2)}\right) = 2\tan^{-1}x$$

Hence we have to find $y_n$ if $y = 2\tan^{-1}x$
Now, $y = 2\tan^{-1}x$

$$y_1 = \frac{2}{1+x^2} = \frac{2}{2i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right]$$

$$= \frac{1}{i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right]$$

Hence,

$$y_n = \frac{1}{i}(-1)^{n-1}\underline{|(n-1)}\left[\frac{1}{(x-i)^n} - \frac{1}{(x+i)^n}\right]$$

Suppose x = r cosθ and 1 = r sinθ so that

$r^2=x^2+1$ and $\theta = \tan^{-1}\left(\frac{1}{x}\right)$

Thus,

$$y_n = \frac{(-1)^{n-1}\underline{|n-1}}{ir^n}\left\{(\cos\theta - i\sin\theta)^{-n} - (\cos\theta + i\sin\theta)^{-n}\right\}$$

$$\Rightarrow y_n = \frac{(-1)^{n-1}\underline{|n-1}}{i.r^n}.2i\sin n\theta$$

$$\Rightarrow y_n = 2(-1)^{n-1}\underline{|(n-1)}.\sin n\theta.\sin^n\theta$$

Where $\theta = \tan^{-1}\left(\frac{1}{x}\right)$ Answer

**Example 4 :** If $y = x\log\frac{x-1}{x+1}$, show that

$$y_n = (-1)^{n-2}\underline{|(n-2)}\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]$$

(U.P.T.U 2003)

**Solution :-**

y = x[ log(x-1) - log (x+1) ] (i)

Differentiating y w.r.t x, we get

$$y_1 = x\left(\frac{1}{x-1} - \frac{1}{x+1}\right) + \log(x-1) - \log(x+1)$$

$$= \frac{1}{x-1} + \frac{1}{x+1} + \log(x-1) - \log(x+1) \qquad \text{(ii)}$$

Again, Differentiating (ii), (n-1) times w.r.t. x we gets

$$y_n = \frac{(-1)^{n-1}\underline{|(n-1)}}{(x-1)^n} + \frac{(-1)^{n-1}\underline{|(n-1)}}{(x+1)^n} + \frac{(-1)^{n-2}\underline{|(n-2)}}{(x-1)^{n-1}} - \frac{(-1)^{n-2}\underline{|(n-2)}}{(x+1)^{n-1}}$$

$$= (-1)^{n-2}\underline{|(n-2)}\left[\frac{(-1)(n-1)}{(x-1)^n} + \frac{(-1)(n-1)}{(x+1)^n} + \frac{x-1}{(x-1)^n} - \frac{x+1}{(x+1)^n}\right]$$

$= (-1)^{n-2}\underline{|(n-2)}\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]$ Answer

**Leibnitz's Theorem :-**

**Statement :-** If u and v are functions of x then

$D^n(uv) = (D^n u).v + {}^nC_1 D^{n-1} u.\, Dv + {}^nC_2 D^{n-2} u.\, D^2v + \ldots\ldots\ldots\ldots + {}^nC_r D^{n-r} u.\, D^rv + \ldots\ldots\ldots\ldots + u.D^nu$

**Proof :-** We shall prove this theorem by mathematical induction method. By differentiation, we have

$D(uv) = (Du).\, v + u.\, (Dv)$

$= (Du).\, v + ({}^1C_1\, u)\, (Dv)$

Thus the theorem is true for n = 1.

Let the theorem be true for n = m i.e.

$D^m(uv) = (D^m u).\, v + {}^mC_1 D^{m-1} u.\, Dv + {}^mC_2 D^{m-2} u.\, D^2v + \ldots\ldots\ldots + m_{C_r} D^{m-r}u.\, D^rv + \ldots\ldots\ldots\ldots\ldots\ldots + uD^mv$

Differentiating the above once again, we get

$D^{m+1}(uv) = (D^{m+1}u).\, v + D^mu.\, Dv + {}^mC_1\{D^mu.\, Dv + D^{m-1}u.\, D^2v\} + {}^mC_2\{D^{m-1}u.D^2v + D^{m-2}u.\, D^3v\} + \ldots\ldots\ldots\ldots + {}^mC_r\{D^{m-r+1}u.\, D^rv + D^{m-r}u.\, D^{r+1}v\} + \ldots\ldots\ldots\ldots + \ldots\ldots\ldots + \{Du.\, D^mv + uD^{m+1}v\}$

Rearranging the terms, we get

$D^{m+1}(uv) = (D^{m+1}u)v + (1 + {}^mC_1)\, D^mu.Dv + ({}^mC_1 + {}^mC_2)\, D^{m-1}u.\, D^2v + \ldots\ldots\ldots\ldots + ({}^mC_r + {}^mC_{r+1})\, D^{m-r}u.\, D^{r+1}v + \ldots\ldots\ldots\ldots + u.\, D^{m+1}v.$

Now using ${}^mC_r + {}^mC_{r+1} = {}^{m+1}C_{r+1}$, we have

$D^{m+1}(uv) = (D^{m+1}u).v + {}^{m+1}C_1 D^mu.Dv + \ldots\ldots\ldots + {}^{m+1}C_{r+1}D^{m-r}u.D^{r+1}\, v + \ldots\ldots\ldots + u.D^{m+1}v$

Thus we have seen that if the theorem is true for n = m, it is also true for n = m+1. Therefore by principle of induction, the theorem is true for every positive integral value of n.

**Example 5 :** If $y = \cos(m \sin^{-1} x)$ prove that

$(1-x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} + (m^2 - n^2)y_n = 0$

**Solution :** $y = \cos(m \sin^{-1} x)$

$\therefore y_1 = -\sin(m \sin^{-1}x).\, m\, \frac{1}{\sqrt{1-x^2}}$

or $(1-x^2)\, y_1^2 = m^2 \sin^2(m \sin^{-1} x)$

$= m^2\, [1 - \cos^2(m \sin^{-1} x)]$

$= m^2 - m^2 \cos^2(m \sin^{-1} x)$

$\therefore\ (1 - x^2)\, y_1^2 = m^2 - m^2y^2$

or $(1-x^2)\, y_1^2 + m^2y^2 - m^2 = 0$

Differentiating again, we have $(1-x^2).\, 2y_1y_2 + y_1^2\, (-2x) + m^2 .\, 2yy_1 = 0$

or $(1 - x^2)\, y_2 - xy_1 + m^2y = 0$

Differentiating n times by Leibnitz's theorem, we get

$y_{n+2}(1-x^2) + {}^nC_1\, y_{n+1}(-2x) + {}^nC_2\, y_n(-2) - \{y_{n+1}.\, x + n_{C_1}\, y_n.1\} + m^2 y_n = 0$

or $(1-x^2)y_{n+2} - 2nx\, y_{n+1} - n(n-1)y_n - xy_{\,n+1} - ny_n + m^2 y_n = 0$

or $(1-x^2)y_{n+2} - (2n+1)\, xy_{\,n+1} + (m^2-n^2) = 0$ Hence proved.

**Example 6 :** If $y = e^{asin^{-1}x}$, show that $(1-x^2)\, y_{n+2} - (2n+1)xy_{\,n+1} - (n^2+a^2)\, y_n = 0$

**Solution :** Given $y = e^{asin-1x}$ (i)

Differentiating,

$$y_1 = e^{asin^{-1}x}\, a\, \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{ay}{\sqrt{(1-x^2)}} \tag{ii}$$

Squaring and cross-multiplying, we get

$y_1^2(1-x^2) = a^2y^2$

Differentiating both sides with respect to x, we get

$y_1^2(-2x) + 2y_1y_2(1-x^2) = a^2 .\, 2yy_1$

or $(1-x^2)y_2 - xy_1 - a^2y = 0$ (iii)

Differentiating each term of this equation n times with respect to x, we get

$D^n[(1-x^2)y_2] - D^n(x\, y_1) - a^2\, D^n(y) = 0$

or $[(1-x^2)y_{n+2} + ny_{n+1}(-2x) + \dfrac{n(n-1)}{1,2}\, y_n\,(-2)] - [xy_{n+1} + ny_n.1] - a^2y_n = 0$

or $(1-x^2)\, y_{n+2} - (2x+1)\, xy_{n+1} - (n^2 + a^2)\, y_n = 0$

Hence proved.

**Example 7:** If $y = (x^2-1)^n$, Prove that

$(x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$

**Solution :** Given $y = (x^2-1)^n$

Differentiating, $y_1 = n(x^2-1)^{n-1}.\, 2x$

or $y_1(x^2-1) = n(x^2-1)^{n-1}.\, 2x$

$= ny.\, 2x$

$= 2n\, xy$

Differentiating again both sides with respect to x, we get

$y_1(2x) + y_2\,(x^2-1) = 2n(xy_1+y)$

or $(x^2-1)\, y_2 + 2x(1-n)y_1 - 2ny = 0$

Differentiating each term of this equation n times with respect to x (by Leibnitz's theorem), we get

$D^n[(x^2-1)y_2] + D^n[2x(1-n)y] - 2nD^ny = 0$

or $[(x^2-1)y_{n+2} + ny_{n+1}\,(2x) + \dfrac{n(n-1)}{\underline{|2}}\, y_n(2)] + 2(1-n)[xy_{n+1} + ny_n] - 2ny_n = 0$

or $(x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$ Hence proved

**Example 8 :** If $\cos^{-1}\left(\frac{y}{b}\right)=\log\left(\frac{x}{n}\right)^n$. Prove that

$x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0$

**Solution :** Given $\cos^{-1}(y/b) = \log(x/n)^n$

or $\cos^{-1}(y/b) = n\log(x/n)$

or $y = b\cos\{n\log(x/n)\}$

Differentiating,

$y_1 = -b\sin\{n\log(x/n)\}\, n.\frac{1}{(x/n)}.\frac{1}{n}$

or $xy_1 = -b\,n\sin\{n\log(x/n)\}$

Again differentiating both sides with respect to x, we get

$xy_2 + y_1 = -bn\cos\{n\log\left(\frac{x}{n}\right)\}\, n.\frac{1}{(x/n)}.\frac{1}{n}$

or $x^2y_2 + xy_1 = -n^2b\cos\{n\log(x/n)\}$

$= -n^2y$, from (i)

or $x^2y_2 + xy_1 + n^2y = 0$

Differentiating each term of this equation n terms, with respect to x, we get

$D^n(x^2y_2)+D^n(xy_1)+n^2D^n(y) = 0$

or $y_{n+2}x^2 + ny_{n+1}(2x) + \frac{n(n-1)}{1.2}y_n(2) + (y_{n+1}x + ny_n) + n^2y_n = 0$

or $x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0$ Hence proved

**Example 9 :** If $x = \cos h\left[\left(\frac{1}{m}\right)\log y\right]$ Prove that

$(x^2-1)y_2 + xy_1 - m^2y = 0$

and $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$

**Solution :** Here $x = \cos h\left[\left(\frac{1}{m}\right)\log y\right]$

or $\cos h^{-1}x = \frac{1}{m}\log y$

or $m\cos h^{-1}x = \log y$

or $y = e^{m\cos h^{-1}x}$ (i)

$\therefore y_1 = e^{m\cos h^{-1}x}\, m\left[\frac{1}{\sqrt{(x^2-1)}}\right]$

or $y_1\sqrt{(x^2-1)} = my$

or $(x^2-1)y_1^2 = m^2y^2$ (ii)

Again differentiating (ii) we get

$(x^2-1)\,2y_1 y_2 + (2x)y_1^2 = m^2.\,2yy_1$

or $(x^2-1)\,y_2 + xy_1 - m^2y = 0$ (iii)

Differentiating (iii) n times by Leibnitz's theorem, we get

$$[(x^2-1)y_{n+2} + n\,(2x)y_{n+2} + \frac{n(n-1)}{\lfloor 2}\,(2)\,y_n] + [xy_{n+1} + n(1)y_n] - m^2\,y_n = 0$$

or $(x^2-1)\,y_{n+2} + (2n+1)\,xy_{n+1} + (n^2 - m^2)\,y_n = 0$ Hence Proved

**Example 10 :** If $y^{1/m} + y^{-1/m} = 2x$ Prove that $(x^2-1)\,y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2) = 0$

**Solution :** Given $y^{1/m} + \dfrac{1}{y^{1/m}} = 2x$

or $y^{2/m} - 2xy^{1/m} + 1 = 0$

or $y^{1/m} = \dfrac{2x \pm \sqrt{(4x^2-4)}}{2}$

or $y^{1/m} = x \pm \sqrt{(x^2-1)}$

or $y = \left[x \pm \sqrt{(x^2-1)}\right]^m$ (i)

when $y = \left[x + \sqrt{(x^2-1)}\right]^m$, we have

$$y_1 = m\left[x + \sqrt{(x^2-1)}\right]^{m-1}\left[1 + \frac{2x}{2\sqrt{(x^2-1)}}\right]$$

or $y_1\sqrt{(x^2-1)} = my$

or $y_1^2\,(x^2-1) = m^2y^2$ (ii)

when $y = \left[x - \sqrt{(x^2-1)}\right]^m$, we have

$$y_1 = m\left[x - \sqrt{(x^2-1)}\right]^{m-1}\left[1 - \frac{2x}{2\sqrt{(x^2-1)}}\right]$$

$$= \frac{-m\left[x - \sqrt{(x^2-1)}\right]^m}{\sqrt{x^2-1}}$$

or $y_1\sqrt{(x^2-1)} = -\,my$

or $y_1^2(x^2-1) = m^2y^2$ (iii)

which is the same result as (ii).

Hence for both the values of y given by (i) we get $y_1^2\,(x^2-1) = m^2y^2$

Differentiating both sides of this with respect to x, we get

$y_1^2\,(2x) + 2y_1y_2\,(x^2-1) = m^2.2yy_1$

or $y_2\,(x^2-1) + xy_1 - m^2y = 0$

Differentiating each term of this equation n times (by Leibnitz's theorem) with respect to x, we get

$D^n\{y_2(x^2-1)\}+D^n(xy_1) - m^2 D^n(y) =0$

or $\{(x^2 -1)\, y_{n+2} + ny_{n+1}\,(2x) + \frac{n(n-1)}{1.2}\, y_n\,(2)\} + \{xy_{n+1} + ny_n.1\} - m^2\, y_n =0$

or $(x^2 -1)\, y_{n+2} + (2n +1)\, xy_{n+1} + (n^2 - m^2)\, y_n =0$ Hence Proved

**Example 11 :** If $y = (x^2 -1)^n$ Prove that $(x^2 -1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n =0$

Hence If $P_n = \frac{d^n}{dx^n}(x^2 - 1)^n$, show that

$\frac{d}{dx}\left\{\left(1-x^2\right)\frac{dP_n}{dx}\right\} +n\,(n+1)P_n =0$

**Solution :** Given $y = (x^2 -1)^n$

Differentiating

$y_1 = n(x^2 -1)^{n-1}.\, 2x$

or $y_1\,(x^2 -1) = n\,(x^2 -1)^n\, 2x$

or $y_1(x^2 -1) = ny.2x$

$= 2nxy$

Differentiating again both sides with respect to x, we get

$y_1(2x) +y_2\,(x^2 -1) = 2n\,[xy_1 +y]$

or $(x^2 -1)\, y_2 + 2x(1 - n)\, y_1 - 2ny =0$

Differentiating each term of this equation n times with respect to x (by Leibnitz's theorem), we get

$D^n\{(x^2 -1)y_2\} +D^n\{2x(1 - n)y\} -2nD^n\, y) =0$

or $[(x^2-1)y_{n+2} + ny_{n+1}\,(2x) + \frac{n(n-1)}{1.2}\, y_n(2)] + 2(1 - n)\{xy_{n+1} + ny_n\} -2ny_n =0$

or $(x^2 -1)y_{n+2} + 2xy_{n+1} - n\,(n+1)y_n =0$ (i)

which Proves the first part.

Again $P_n = \frac{d^n}{dx^n}(x^2-1)^n$

$= \frac{d^n}{dx^n}(y)$

$= y_n$

$\therefore \frac{d}{dx}\left\{\left(1-x^2\right)\frac{dP_n}{dx}\right\} = \frac{d}{dx}\left\{\left(1-x^2\right).y_{n+1}\right\}$

$\because P_n = y_n$

$= (1-x^2)y_{n+2} - 2xy_{n+1}$

$= -[(x^2 -1)y_{n+2} + 2x\, y_{n+1}]$

$= - n(n+1)y_n$ from (i)

$= - n\,(n+1)P_n$ $\because P_n = y_n$

or $\frac{d}{dx}\left\{\left(1-x^2\right)\frac{dP_n}{dx}\right\}$ + n(n+1) $P_n$ =0. Hence proved

**Problem 12 :** If y = $\sin^{-1}x$ , find $(y_n)_0$

(U.P.T.U. 2009)

**Solution :** Given y = $\sin^{-1}x$ (i)

Differentiating, $y_1 = \frac{1}{\sqrt{(1-x^2)}}$ (ii)

or $y_1^2(1-x^2)=1$

Differentiating both sides again w.r.t x we get

$2y_1y_2(1-x^2)+y_1^2(-2x)=0$

or $y_2(1-x^2)-xy_1=0$.........(iii)

Differentiating both sides of (iii) n times with respect to x by Leibnitz's theorem, we get

$[y_{n+2}(1-x^2)+n.y_{n+1}(-2x)+\frac{n(n-1)}{\lfloor 2}y_n(-2)]-[xy_{n+1}+n.1.\,y_n]=0$

or $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0$ (iv)

putting x = 0 in (i), (ii), (iii) and (iv) we get

$(y)_0=\sin^{-1}0=0$; $(y_1)_0=1$; $(y_2)_0=0$

and $(y_{n+2})_0=n^2(y_n)_0$ (v)

If n is even, putting n = 2, 4, 6............, we get

$(y_4)_0=2^2(y_2)_0=0$ $\therefore (y_2)_0=0$

$(y_6)_0=4^2(y_4)_0=0$, $\therefore (y_4)_0=0$

In this way we can show that for all even values of n, $(y_n)_0=0$ Answer

If n is odd, putting n=1,3,5,7............(n-2) in (v) we get

$(y_3)_0=(y_1)_0=1$; $(y_5)_0=3^2(y_3)_0=3^2.1$, $(y_7)_0=5^2.(y_5)_0=5^2.\,3^2.1$ etc

$(y_n)_0=(n-2)^2(y_{n-2})_0$

$=(n-2)^2.\,(n-4)^2(y_{n-4})_0$

$=(n-2)(n-4)^2$..........$5^2, 3^2, 1^2$. Answer.

**Example 13 :** If y = $[\log\{x+\sqrt{(1+x^2)}\}]^2$, show that

$(y_{n+2})_0=-n^2(y_n)_0$, hence find $(y_n)_0$.

**Solution :** Let y = $[\log x+\sqrt{(1+x^2)}]^2$ (i)

$$\therefore y_1=2\left[\log\left\{x+\sqrt{(1+x^2)}\right\}\right]\frac{1}{\left\{x+\sqrt{(1+x^2)}\right\}}\left[1+\frac{2x}{2\sqrt{1+x^2}}\right]$$

$$\text{or } y_1=\left[\frac{2}{\sqrt{(1+x^2)}}\right]\left[\log\left\{x+\sqrt{(1+x^2)}\right\}\right]$$

or $y_1^2(1+x^2) = 4[\log\{x+\sqrt{(1+x^2)}\}]^2$

or $y_1^2(1+x^2)= 4y$ (ii)

Again differentiating,

$y_1^2(2x) + (1+x^2)2y_1y_2 = 4y_1$

or $y^2(1+x^2) + xy_1 - 2 = 0$ (iii)

Differentiating each term of this equation n times, by Leibnitz's theorem, with respect to x, we have

$$\{y_{n+2}(1+x^2) + ny_{n+1}(2x)+\frac{n(n-1)}{1.2}y_{n}.2\} + \{y_{n+1}(x) + ny_n(1)\} = 0$$

or $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$ ...........(iv)

putting x =0 in (iv) we have

$(1+0)(y_{n+2})_0 + n^2(y_n)_0 = 0$

or $(y_{n+2})_0 = -n^2(y_n)_0$.................(v)

Putting n = n-2, n-4, n-6..........in (v), we get

$(y_n)_0 = -(n-2)^2(y_{n-2})_0$;

$(y_{n-2})_0 = -(n-4)^2(y_{n-4})_0$;

$(y_{n-4})_0 = -(n-6)^2(y_{n-6})_0$;

where we have (on multiplying side wise)

$(y_n)_0 = \{-(n-2)^2\}\{-(n-4)^2\}\{-(n-6)^2\}\{(y_{n-6})_0\}$

∴ If n is odd,

$(y_n)_0 = \{-(n-2)^2\}\{-(n-4)^2\}\{-(n-6)^2\}$..................$\{-3^2\}\{-1^2\}(y_1)_0$

Now from (i) putting x =0, we get

$(y)_0 = [\log(0+1)]^2 = (\log 1)^2 = 0$

∴ From (ii) putting x =0, we get

$(y_1)_0^2 = 4(y)_0 = 0$ or $(y_1)_0 = 0$

Hence when n is odd $(y_n)_0 = 0$ Answer

And if n is even, as before

$(y_n)_0 = \{-1(n-2)^2\}\{-(n-4)^2\}\{-(n-6)^2\}$................$\{-4^2\}\{-2^2\}(y_2)_0$

Also from (iii) putting x =0, we get

$(y_2)_0(1+0) + 0 - 2 = 0$

or $(y_2)_0 = 0$

∴ where n is even, we have

$(y_n)_0 = \{-(n-2)^2\}\{-(n-4)^2\}\{-(n-6)^2\}$.............$(-2)^2\}2$

$= (-1)^{(n-2)/2}(n-2)^2(n-4)^2(n-6)$.......$4^2.2^2.2$ Answer

**Example 14 :** If $y = \left[x+\sqrt{(1+x^2)}\right]^m$, find $(y_n)_0$

**Solution :** Given $y = \left[x+\sqrt{(1+x^2)}\right]^m$ ...........(i)

$$\therefore y_1 = m\left[x+\sqrt{(1+x^2)}\right]^{m-1}\left[1+\frac{x}{\sqrt{1+x^2}}\right]$$

$$= m\left[x+\sqrt{(1+x^2)}\right]^{m-1}\left[\frac{\sqrt{(1+x^2)}+x}{\sqrt{(1+x^2)}}\right]$$

$$= \frac{m\left[x+\sqrt{(1+x^2)}\right]^{m}}{\sqrt{(1+x^2)}}$$

$$= \frac{my}{\sqrt{(1+x^2)}}$$

Again differentiating

$y_1^2\,(2x) + (1+x^2)\,2y_1y_2 = m^2.\,2yy_1$

or $y_2\,(1+x^2) + xy_1 - m^2y = 0$……………….(iii)

Differentiating this equation n time by Leibnitz's theorem, we get

$[y_{n+2}\,(1+x^2) + ny_{n+1}\,(2x) + \frac{n(n-1)}{1.2}\,y_n\,(2)] + [xy_{n+1} + ny_n] - m^2\,y_n = 0$

or $(1+x^2)y_{n+2} + (2n+1)\,xy_{n+1} + (n^2 - m^2)y_n = 0$

putting x =0 we get

$(y_{n+2})_0 = (m^2 - n^2)\,(y_n)_0$……………………….(iv)

If n is odd, putting n = 1, 3, 5, 7………….(n-2), we get

When n = 1, $(y_3)_0 = (m^2 - 1^2)\,(y_1)_0$

And from (i) and (ii), putting x =0, we get

$(y)_0 = 1;\ (y_1)_0^2 = m^2\,(y)_0^2 = m^2$ or $(y_1)_0 = m$

$\therefore\ (y_3)\,0 = (m^2-1^2)\,m$

when n = 3, from (iv) we get $(y_5)_0 = (m^2 - 3^2)(y_3)_0$

or $(y_5)_0 = (m^2 - 3^2)\,(m^2 - 1)\,m$

when n =5, from (iv) we get $(y_7)_0 = (m^2 - 5^2)\,(y_5)_0$

or $(y_7)_0 = (m^2 - 5^2)\,(m^2 - 3^2)\,(m^2 - 1^2)m.$

Proceeding in this manner, when n = n - 2 from (iv) we get

$(y_n)_0 = \{m^2 - (n-2)^2\}\,(y_{n-2})_0$

$= \{m^2 - (n-2)^2\}\,\{m^2 - (n-4)^2\}\,\{m^2 - (n-6)^2\}$…………….$(m^2 - 3^2)\,(m^2 - 1^2)\,m$ Answer.

If n is even, putting n = 2, 4, 6, 8,…………….(n-2) is (iv), we get

when n =2, $(y_4)_0 = (m^2 - 2^2)\,(y_2)_0$

And from (iii) putting x =0, we get

$(y_2)_0 + 0 + m^2\,(y)_0 = 0$ or $(y_2)_0 = m^2\,(y)_0 = m^2$

$\therefore (y_4)_0 = (m^2 - 2^2)m^2$

When n =4, from (iv) we get $(y_6)_0 = (m^2 - 4^2)\,(y_4)_0$

or $(y_6)_0 = (m^2 - 4^2)\,(m^2 - 2^2)\,m^2$

when n =6, from (iv), we get $(y_8)_0 = (m^2 - 6^2)\,(y_6)_0$
or $(y_8)_0 = (m^2 - 6^2)\,(m^2 - 4^2)\,(m^2 - 2^2)m^2$
Proceeding in this manner, when n = n-2 from (iv) we have
$(y_n)_0 = \{m^2 - (n-2)^2\}\,(y_{n-2})_0$
$= \{m^2 - (n-2)^2\}\,\{m^2 - (n-4)^2\}\ldots\ldots\ldots\ldots(m^2 - 4^2)\,(m^2 - 2^2)m^2.$
Answer

## Exercise

1. If $e^{m\cos^{-1}x}$, prove that
   $(1-x^2)\,y_{n+2} = x\,(2n+1)\,y_{n+1} - (n^2 + m^2)\,y_n = 0$
2. If y = sin (a sin⁻¹ x) prove that
   $(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - (n^2 - a^2)\,y_n = 0$

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3. If y = a cos (log x) + b sin (log x) show that
   $x^2y_{n+2} + (2n+1)\,xy_{n+1}\,(n^2+1)\,y_n = 0$

   (U.P.T.U 2004)
4. If y = sin (m sin⁻¹ x) prove that
   $(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} + (m^2 - n^2)y_n = 0$

   (U.P.T.U 2003. 05)
5. If $y = e^{m\cos^{1}x}$, show that
   $(1-x^2)y_{n+2} - (2n+1)\,xy_{n+1} - (m^2+n^2)\,y_n = 0$
   and hence evaluate $(y_n)0$

Ans: $(y_n)_0 = -\{(n-2)^2 + m^2\}\,\{(n-4)^2 + m^2\}\ldots\ldots\ldots\{(1^2+m^2)\,me\frac{m\pi}{2}\}$, for odd values of n

$(y_n)_0 = \{(n-2)^2 + m^2\}\,\{(n-4)^2 + m^2\}\ldots\ldots\ldots\ldots\{m^2e\frac{m\pi}{2}\}$ for even values of n.

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# Chapter 2

# Partial Differentiation

**Partial differential coefficients :** The Partial differential coefficient of f(x,y) with respect to x is the ordinary differential coefficient of f(x,y) when y is regarded as a constant. It is written as

$\frac{\partial f}{\partial x}$ or $\partial f/\partial x$ or $D_x f$

Thus $\frac{\partial f}{\partial x} = \lim\limits_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$

Again, the partial differential coefficient $\partial f/\partial y$ of f(x,y) with respect to y is the ordinary differential coefficient of f(x,y) when x is regarded as a constant.

Thus $\frac{\partial f}{\partial y} = \lim\limits_{k \to 0} \frac{f(x.y+k) - f(x,y)}{k}$

Similarly, if f is a function of the n variables $x_1$, $x_2$,.........$x_n$, the partial differential coefficient of f with respect to $x_1$ is the ordinary differential coefficient of f when all the variables except $x_1$ are regarded as constants and is written as $\partial f/\partial x_1$. $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are also denoted by $f_x$ and $f_y$ respectively.

The partial differential coefficients of $f_x$ and $f_y$ are $f_{xx}$, $f_{xy}$, $f_{yx}$, $f_{yy}$

or $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2}$, respectively.

It should be specially noted that $\frac{\partial^2 f}{\partial y \partial x}$ means $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ and $\frac{\partial^2 f}{\partial x \partial y}$ means $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$.

The student will be able to convince himself that in all ordinary cases

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

**Example 1 :** If u = log ($x^3 + y^3 + z^3 - 3xyz$) show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+x)^2}$$

(U.P.T.U. 2004, B.P.S.C. 2007)
(U.P.P.C.S. 2003)

**Solution :** The given relation is

u = log($x^3 + y^3 + x^3 - 3xyz$)

Differentiate it w.r.t. x partially, we get

$$\frac{\partial u}{\partial x}=\frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

similarly $\frac{\partial u}{\partial y}=\frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$

and $\frac{\partial u}{\partial z}=\frac{3z^2-3xyz}{x^3+y^3+z^3-3xyz}$

$$\therefore \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3(x^2+y^2+z^2-yz-zx-xy)}{x^3+y^3+z^3-3xyz}$$

$$=\frac{3\left(x^2+y^2+z^2-yz-xz-xy\right)}{(x+y+z)\left(x^2+y^2+z^2-y^z-zx-xy\right)}$$

$$=\frac{3}{x+y+z}$$

Now $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$=3\left[\frac{\partial}{\partial x}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{1}{x+y+z}\right)\right]$$

$$=3\left[-\frac{1}{(x+y+z)^2}+\frac{-1}{(x+y+z)^2}+\frac{-1}{(x+y+z)^2}\right]$$

$$=3\left[\frac{-3}{(x+y+z)^2}\right]$$

$$=-\frac{9}{(x+y+z)^2}$$ Hence Proved.

**Example 2:** If u = $e^{xyz}$, show that

$$\frac{\partial^3 u}{\partial x\partial y\partial z}=(1+3xyz+x^2y^2z^2)\,e^{xyz}$$

(U.P.P.C.S. 1998, B.P.S.C 2005)

**Solution :** Given $u = e^{xyz}$

$\therefore \frac{\partial u}{\partial z} = xy\, e^{xyz}$

$\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}\left(xy\, e^{xyz}\right) = x \frac{\partial}{\partial y} y e^{xyz}$

$= x[y\, xz\, e^{xyz} + e^{xyz}]$

$= e^{xyz}\, (x^2yz + x)$

Hence $\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\, [e^{xyz}\, (x^2yz+x)]$

$= e^{xyz}\, (2xyz+1) + yz\, e^{xyz}\, (x^2yz+x)$

$= e^{xyz}\, [2xyz + 1 + x^2y^2z^2 + xyz]$

$= e^{xyz}\, (1+3xyz + x^2y^2z^2)$ Hence Proved.

**Example 3 :** If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, prove that,

$u_x^2 + u_y^2 + u_z^2 = 2\, (xu_x + yu_y + zu_z)$

(U.P.T.U. 2003)

**Solution :** Given $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$ .................(i)

where u is a function of x,y and z,

Differentiating (i) partially with respect to x, we get

$$\frac{(a^2+u).2x - x^2 \frac{\partial u}{\partial x}}{\left(a^2+u\right)^2} + \frac{(b^2+u).0 - y^2 \frac{\partial u}{\partial x}}{\left(b^2+u\right)^2} + \frac{(c^2+u).0 - z^2 \frac{\partial u}{\partial x}}{\left(c^2+u\right)^2} = 0$$

or $\frac{2x}{a^2+u} - \left[\frac{x^2}{\left(a^2+u\right)^2} + \frac{y^2}{\left(b^2+u\right)^2} + \frac{z^2}{\left(c^2+u\right)^2}\right] \frac{\partial u}{\partial x} = 0$

or $\frac{\partial u}{\partial x} = \frac{2x/\left(a^2+u\right)}{\left[x^2/\left(a^2+u\right)^2 + y^2/\left(b^2+u\right)^2 + z^2/\left(c^2+u\right)^2\right]}$

$= \frac{2x/a^2+u}{\sum\left[x^2/\left(a^2+u\right)^2\right]}$

Similarly $\frac{\partial u}{\partial y} = \frac{2y/\left(b^2+u\right)}{\sum\left[x^2/\left(a^2+u\right)^2\right]}$

and $\frac{\partial u}{\partial z} = \frac{2z/\left(c^2+u\right)}{\sum\left[x^2/\left(a^2+u\right)^2\right]}$

$$\therefore \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2 = 4\frac{\left[x^2/\left(a^2+u\right)^2+y^2/\left(b^2+u\right)^2+z^2/\left(c^2+u\right)^2\right]}{\left[\sum\left\{x^2/\left(a^2+u\right)^2\right\}\right]^2}$$

or $u_x^2+u_y^2+u_z^2 = \dfrac{4}{\sum\left[\left\{x^2/\left(a^2+u\right)^2\right\}\right]}$ ....................(ii)

Also $xu_x + yu_y + zu_z = = x\left(\dfrac{\partial u}{\partial x}\right)+y\left(\dfrac{\partial u}{\partial y}\right)+z\left(\dfrac{\partial u}{\partial z}\right)$

$$=\frac{1}{\sum\left[x^2/\left(a^2+u\right)^2\right]}\left[\frac{2x^2}{\left(a^2+u\right)}+\frac{2y^2}{\left(b^2+u\right)}+\frac{2z^2}{\left(c^2+u\right)}\right]$$

$=\dfrac{2}{\sum\left[x^2/\left(a^2+u\right)^2\right]}[1]$ .................(iii)

From (i), (ii) (iii) and we have

$u_x^2+u_y^2+u_z^2 = 2(xu_x+yu_y+zu_z)$ Hence Proved.

**Example 4 :** If u = f(r) and x = r cosθ, y = r sinθ i.e. $r^2 = x^2+y^2$, Prove that

$\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2} = f''(r)+\dfrac{1}{r}\,f'(r)$

(U.P.T.U. 2000, 2005)

**Solution :** Given u = f(r)...............(i)

Differentiating (i) partially w.r.t. x, we get

$\dfrac{\partial u}{\partial x} = f'(r).\dfrac{\partial r}{\partial x}$

$= f'(r).\,\dfrac{x}{r}$ $\qquad \because r^2 = x^2+y^2$

$\Rightarrow 2r\,\dfrac{\partial r}{\partial x} = 2x$

$\Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

Similarly $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$

Differentiating above once again, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{xf'(r)}{r}\right]$$

$$= \frac{r\left[f'(r).1 + xf''(r)(\partial r/\partial x)\right] - xf'(r)(\partial r/\partial x)}{r^2}$$

or $\frac{\partial^2 u}{\partial x^2} = \frac{1}{r^2}$ [rf' (r) + $x^2$ f''(r) - $\frac{x^2}{r}$ f'(r)] (ii)

Similarly, $\frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2}$ [rf'(r) + $y^2$ f''(r) - $\frac{y^2}{r}$ f'(r)] (iii)

Adding (ii) and (iii), we get

$$\frac{\partial u^2}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2}\left[2rf'(r) + \left(x^2 + y^2\right)f''(r) - \frac{\left(x^2 + y^2\right)}{r}f'(r)\right]$$

$= \frac{1}{r^2}$ [2r f'(r) + $r^2$ f''(r) - r f'(r)]

$= \frac{1}{r}$ f'(r) + f''(r), Hence proved.

**Example 5 :** If $x^x y^y z^z = c$, show that at x = y =z,

$$\frac{\partial^2 z}{\partial x \partial y} = -\left(x \log ex\right)^{-1}$$

(U.P.P.C.S. 1994; P.T.U. 1999)

**Solution :** Given $x^x y^y z^z = c$, where z is a function of x and y

Taking logarithms, x log x + y log y + z log z = log c (i)

Differentiating (i) partially with respect to x, we get

$$\left[x\left(\frac{1}{x}\right) + (\log x)1\right] + \left[z\left(\frac{1}{z}\right) + (\log z)1\right]\frac{\partial z}{\partial x} = 0$$

or $\frac{\partial z}{\partial x} = -\frac{(1 + \log x)}{(1 + \log z)}$ (ii)

Similarly from (i) we have

$\frac{\partial z}{\partial y} = -\frac{(1 + \log y)}{(1 + \log z)}$ (iii)

$$\therefore \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$= \frac{\partial}{\partial x}\left[-\left(\frac{1 + \log y}{1 + \log z}\right)\right]$ From (iii) or $\frac{\partial^2 z}{\partial x \partial y} = -(1+\log y).\frac{\partial}{\partial x}\left[(1+\log z)^{-1}\right]$

$= -(1+\log y).\left[-(1+\log z)^{-2}.\frac{1}{z}.\frac{\partial z}{\partial x}\right]$

or $\frac{\partial^2 z}{\partial x \partial y} = \frac{(1+\log y)}{z(1+\log z)^2}.\left\{-\left(\frac{1+\log x}{1+\log z}\right)\right\}$, using (ii)

At x = y = z, we have $\frac{\partial^2 z}{\partial x \partial y} = -\frac{(1+\log x)^2}{x(1+\log x)^3}$

Substituting x for y and z

i.e. $\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x(1+\log x)}$

$= -\frac{1}{x(\log e + \log x)}$ $\quad\quad \therefore \log e = 1$

$= -\frac{1}{x \log(ex)}$

$= -\{x \log (ex)\}^{-1}$ Hence Proved.

**Example 6 :** If u = $\sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, show that

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

(U.P.T.U. 2006)

**Solution :** We have u = $\sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$ .............(i)

Differentiating (i) partially w.r.t. x and y, we get.

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}.\frac{1}{y} + \frac{1}{1+\left(\frac{y}{x}\right)^2}.\left(\frac{-y}{x^2}\right)$$

$$= \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

or $x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2}$ ..............(ii) and $\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} - \left(\frac{x}{y^2}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x}$

$= -\frac{x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$

or $x\frac{\partial u}{\partial y} = -\frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}$ ................(iii)

on adding (ii) and (iii), we have

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$ Answer.

## Euler's Theorem on Homogeneous Functions :

**Statement :** If f(x,y) is a homogeneous function of x and y of degree n, then

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$

(U.P.T.U. 2006; P.T.U. 2004)

**Proof :** Since f(x,y) is a homogeneous function of degree n, it can be expressed in the form

$f(x,y) = x^n F(y/x)$...................(i)

$\therefore \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\{x^n F(y/x)\} = nx^{n-1} F(y/x) + x^n F'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$

or $x\frac{\partial f}{\partial x} = n\, x^n F\left(\frac{y}{x}\right) - yx^{n-1} F'\left(\frac{y}{x}\right)$.................(ii)

Again from (i), we have

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\{x^n F(y/x)\}$

$= x^n F'(y/x).\ \frac{1}{x}$

or $y\frac{\partial f}{\partial y} = yx^{n-1} F'(y/x)$.............(iii)

Adding (ii) and (iii), we get

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nx^n F(y/x)$

= nf using(i) Hence Proved.

**Note.** In general if f $(x_1, x_2 \ldots\ldots\ldots\ldots x_n)$ be a homogeneous function of degree n, then by Euler's theorem, we have

$$x_1\frac{\partial f}{\partial x_1}+x_2\frac{\partial f}{\partial x_2}+\dots\dots\dots\dots+x_n\frac{\partial f}{\partial x_n}=nf$$

**Example 7 :** If $u=\log\left(\frac{x^2+y^2}{x+y}\right)$, Prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1$$

(U.P.T.U. 2009)

**Solution :** We are given that

$$u=\log\left(\frac{x^2+y^2}{x+y}\right)$$

$$\therefore\ e^u=\frac{x^2+y^2}{x+y}=f(\text{say})$$

Clearly f is a homogeneous function in x and y of degree 2-1 i.e. 1

∴ By Euler's theorem for f, we should have

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=f$$

$$x\frac{\partial}{\partial x}\left(e^u\right)+y\frac{\partial}{\partial y}\left(e^u\right)=e^u \qquad \because\ f=e^u$$

$$\text{or }\ xe^u\frac{\partial u}{\partial x}+ye^u\frac{\partial u}{\partial y}=e^u$$

$$\text{or }\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1$$ Hence Proved.

**Example 8 :** If $u=\sin^{-1}\left\{\frac{x+y}{\sqrt{x}+\sqrt{y}}\right\}$, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u$$

**Solution :** Here $u=\sin^{-1}\left\{\frac{x+y}{\sqrt{x}+\sqrt{y}}\right\}$

$$\Rightarrow \sin u=\frac{x+y}{\sqrt{x}+\sqrt{y}}=f\,(\text{say})$$

Here f is a homogeneous function in x and y of degree $\left(1-\frac{1}{2}\right)$ i.e $\frac{1}{2}$

∴ By Euler's theorem for f, we have

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=\frac{1}{2}f$$

$$\text{or } x\frac{\partial}{\partial x}(\sin u)+y\frac{\partial}{\partial y}(\sin u)=\frac{1}{2}\sin u$$

$$\therefore f=\sin u$$

$$\text{or } x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=\frac{1}{2}\sin u$$

$$\text{or } x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u.$$

Hence Proved

**Example 9 :** If $u=\tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, then prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\sin 2u$$

(U.P.P.C.S. 2005)

**Solution :** Here $\tan u=\frac{x^2+y^2}{x+y}=f$ (say)

Then for $\frac{x^2+y^2}{x+y}$ is a homogeneous function in x and y of degree 2-1 i.e 1.

$\therefore$ By Euler's theorem for f, we have

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=1.f$$

$$\text{or } x\frac{\partial}{\partial x}(\tan u)+y\frac{\partial}{\partial y}(\tan u)=\tan u$$

$\because f=\tan u$

$$\text{or } x\sec^2 u\frac{\partial u}{\partial x}+y\sec^2 u\frac{\partial u}{\partial y}=\tan u$$

$$\text{or } x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{\tan u}{\sec^2 u}=\sin u\cos u=\frac{1}{2}\sin 2u.$$

Hence Proved

**Example 10 :** If u be a homogeneous function of degree n, then prove that

(i) $x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial x}$

(ii) $x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=(n-1)\frac{\partial u}{\partial y}$

(iii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$

(U.P.T.U. 2005; Uttarakhand T. U. 2006)

**Solution :** Since u is a homogenous function of degree n, therefore by Euler's theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu \;\ldots\ldots\ldots\ldots(1)$$

Differentiating (i) partially w.r.t. x, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}.1+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x}$$

$$\text{or } x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x}-\frac{\partial u}{\partial x}$$

$$\text{or } x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial x} \;\ldots\ldots\ldots\ldots(2)$$

which prove the result (i)

Now differentiating (i) partially w.r.t. y, we get

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+1.\frac{\partial u}{\partial y}=n\frac{\partial u}{\partial y}$$

$$\text{or } x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}=n\frac{\partial u}{\partial y}-\frac{\partial u}{\partial y}$$

$$\text{or } x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}=(n-1)\frac{\partial u}{\partial y} \;\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Which proves the result (ii)

Multiplying (2) by x and (3) by y and then adding, we get

$$x\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x\partial y}+xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)\left\{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right\}$$

$$\text{or } x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)\,nu$$

which proves the result (iii). Hence Proved

**Example 11 :** If $u = xf_1\left(\frac{y}{x}\right)+f_2\left(\frac{y}{x}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0$$

(I.A.S. 2006; U.P.P.C.S. 1997; Uttarkhand T.U. 2006)

**Solution :** Let $v = xf_1\left(\frac{y}{x}\right)$ and $w = f_2\left(\frac{y}{x}\right)$

Then u = v + w..............(i)

Now $v = x\, f_1\,(y/x)$ is a homogeneous function in x and y of degree one and we prove that if u is a homogeneous function of x and y of degree n then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)\,u$$

(see Example 11 (iii))

∴ Here we have

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = 1(1-1)\,v = 0\,......................(ii)$$

Also $w = f_2(y/x)$ is a homogeneous function of x and y of degree zero so we have

$$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x\partial y} + y^2\frac{\partial^2 w}{\partial y^2} = 0(0-1)\,w = 0\,...............(iii)$$

Adding (ii) and (iii) we have

$$x^2\frac{\partial^2}{\partial x^2}(u+v) + 2xy\frac{\partial^2}{\partial x\partial y}(v+w) + y^2\frac{\partial^2}{\partial y^2}(v+w) = 0$$

or $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$

∵ from (i) we have u = v + w. Hence Proved

**Example 12 :** If $u = \sin^{-1}\left[\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right]$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + 3\tan u = 0$$

(U.P.T.U. 2003)

**Solution :** Here given $u = \sin^{-1}\left[\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right]$

$$\Rightarrow \sin u = \frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}} = f\text{ (say)}$$

Now here f is a homogeneous function in x, y, z of degree (1 - 4) i.e -3.

Hence, by Euler's theorem

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = -3f$$

$$\text{or } x\frac{\partial}{\partial x}(\sin u)+y\frac{\partial}{\partial y}(\sin u)+z\frac{\partial}{\partial z}(\sin u) = -3\sin u$$

$\therefore f = \sin u$

$$\text{or } x\cos u\,\frac{\partial u}{\partial x}+y\cos u\,\frac{\partial u}{\partial y}+z\cos u\,\frac{\partial u}{\partial z} = -3\sin u$$

$$\text{or } x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = -3\frac{\sin u}{\cos u}$$

$$\text{or } x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}+3\tan u = 0.$$ Hence Proved

**Example 13 :** If $u(x,y,z) = \log(\tan x + \tan y + \tan z)$ Prove that

$$\sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}+\sin 2z\frac{\partial u}{\partial z} = 2$$

(U.P.T.U. 2006)

**Solution :** we have

$u(x,y,z) = \log(\tan x + \tan y + \tan z)$................(i)

Differentiating (i) w.r.t. 'x' partially, we get

$$\frac{\partial u}{\partial x} = \frac{\sec^2 x}{\tan x+\tan y+\tan z} \text{ ......................(ii)}$$

Differentiating (i) w.r.t. 'y' partially we get

$$\frac{\partial u}{\partial y} = \frac{\sec^2 y}{\tan x+\tan y+\tan z} \text{ ..................(iii)}$$

Again differentiating (i) w.r.t 'z' partially we get

$$\frac{\partial u}{\partial z} = \frac{\sec^2 z}{\tan x+\tan y+\tan z} \text{ ................(iv)}$$

Multiplying (ii), (iii) and (iv) by sin 2x, sin 2y and sin 2z respectively and adding them, we get

$$\sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}+\sin 2z\frac{\partial u}{\partial z} = \frac{\sin 2x\sec^2 x+\sin 2y\sec^2 y+\sin 2z\sec^2 z}{\tan x+\tan y+\tan z}$$

$$= \frac{2\sin x\cos x.\sec^2 x+2\sin y\cos y.\sec^2 y+2\sin z\cos z.\sec^2 z}{\tan x+\tan y+\tan z}$$

$$= \frac{2(\tan x+\tan y+\tan z)}{\tan x+\tan y+\tan z}$$

$= 2$

$$\Rightarrow \sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}+\sin 2z\frac{\partial u}{\partial z} = 2.$$ Hence Proved

## Total Differentiation

**Introduction :** In partial differentiation of a function of two or more variables, only one variable varies. But in total differentiation, increments are given in all the variables.

**Total differential Coefficient :** If u = f(x,y)

where x = ϕ(t), and y = Ψ(t) then we can find the value of u in terms of t by substituting from the last two equations in the first equation. Hence we can regard u as a function of the single variable t, and find the ordinary differential coefficient $\frac{du}{dt}$.

Then $\frac{du}{dt}$ is called the total differential coefficient of u, to distinguish it from the partial differential coefficient $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

The problem now is to find $\frac{du}{dt}$ without actually substituting the values of x and y in f(x,y), we can obtain the requisite formula as follows :

Let $\phi\,(t+\iota) = x + h$, $\psi(t+\tau) = y + k$

Then by definition

$$\frac{du}{dt} = \lim_{\tau\to 0} \frac{f(x+h, y+k) - f(x,y)}{\tau}$$

$$= \lim_{\tau\to 0} \left\{ \frac{f(x+h, y+k) - f(x, y+k)}{h} \cdot \frac{h}{\tau} + \frac{f(x, y+k) - f(x,y)}{k} \cdot \frac{k}{\tau} \right\}$$

Now $\lim\limits_{\tau\to 0} \frac{h}{\tau} = \frac{dx}{dt}$ and $\lim\limits_{\tau\to 0} \frac{k}{\tau} = \frac{dy}{dt}$

Also, if k did not depend on h

$$\lim_{h\to 0} \frac{f(x+h, y+k) - f(x, y+k)}{h}$$

would have been equal to

$$\frac{\partial f(x, y+k)}{\partial x}$$

by definition

Moreover, supposing that $\partial f(x,y)/\,\partial x$ is a continuous function of y

$$\lim_{k\to 0} \frac{\partial f(x, y+k)}{\partial x} = \frac{\partial f(x,y)}{\partial x}$$

we shall assume, therefore that

$$\lim_{\tau\to 0} \frac{f(x+h, y+k) - f(x, y+k)}{h} = \frac{\partial f(x,y)}{\partial x}$$

Hence

$$\frac{du}{dt}=\frac{\partial f}{\partial x}.\frac{dx}{dt}+\frac{\partial f}{\partial y}.\frac{dy}{dt}$$

i.e. $\frac{du}{dt}=\frac{\partial u}{\partial x}.\frac{dx}{dt}+\frac{\partial u}{\partial y}.\frac{dy}{dt}$

Similarly, if $u = f(x_1, x_2 \ldots\ldots x_n)$ and $x_1, x_2 \ldots\ldots\ldots\ldots, x_n$ are all functions of t, we can prove that

$$\frac{du}{dt}=\frac{\partial u}{\partial x_1}.\frac{dx_1}{dt}+\frac{\partial u}{\partial x_2}.\frac{dx_2}{dt}+\ldots\ldots\ldots\ldots+\frac{\partial u}{\partial x_n}.\frac{dx_n}{dt}$$

**An important case :** By supposing t to be the same, as x in the formula for two variables, we get the following proposition :

When f(x,y) is a function of x and y, and y is a function of x, the total (i.e., the ordinary) differential coefficient of f with respect to x is given by

$$\frac{df}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}.\frac{dy}{dx}$$

Now, if we have an implicit relation between x and y of the form f(x,y) = C where C is a constant and y is a function of x, the above formula becomes

$$0=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}.\frac{dy}{dx}$$

Which gives the important formula

$$\frac{dy}{dx}=-\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}$$

Again, if f is a function of n variables $x_1, x_2, x_3, \ldots\ldots x_n$, and $x_2, x_3 \ldots\ldots x_n$ are all functions of $x_1$, the total (i.e. the ordinary) differential coefficient of f with respect to $x_1$ is given by

$$\frac{df}{dx_1}=\frac{\partial f}{\partial x_1}+\frac{\partial f}{\partial x_2}.\frac{dx_2}{dx_1}+\frac{\partial f}{\partial x_3}.\frac{dx_3}{dx_1}+\ldots\ldots\ldots\ldots+\frac{\partial f}{\partial x_n}.\frac{dx_n}{dx_1}$$

**Example 14 :** If u = x log xy, where $x^3+y^3+3xy = 1$, find $\frac{du}{dx}$.

(U.P.T.U. 2002, 05)

**Solution :** Given u = x log xy……………..(i)

we know $\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx}$ ……………….(ii)

Now from (i) $\frac{\partial u}{\partial x}=x.\frac{1}{xy}y+\log xy$

$= 1+\log xy$

and $\frac{\partial u}{\partial y} = x\frac{1}{xy}x = \frac{x}{y}$

Again, we are given $x^3+y^3+3xy=1$, whence differentiating, we get

$$3x^2+3y^2\frac{dy}{dx}+3\left(x\frac{dy}{dx}+y.1\right)=0$$

or $\frac{dy}{dx} = -\frac{(x^2+y)}{(y^2+x)}$

Substituting these values in (ii) we get

$$\frac{du}{dx} = (1+\log xy)+\frac{x}{y}\left[-\frac{(x^2+y)}{(y^2+x)}\right] \text{ Answer.}$$

**Example 15 :** If $f(x, y)=0$, $\phi(y, z)=0$ show that

$$\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial z}\frac{dz}{dx} = \frac{\partial f}{\partial x}\frac{\partial \phi}{\partial y}$$

**Solution :** If $f(x, y)=0$ then $\frac{dy}{dx} = -\left(\frac{\partial f}{\partial x}\right)/\left(\frac{\partial f}{\partial y}\right)$............(i)

if $\phi(y, z)=0$, then $\frac{dz}{dy} = -\left(\frac{\partial \phi}{\partial y}\right)/\left(\frac{\partial \phi}{\partial z}\right)$....................(ii)

Multiplying (i) and (ii), we have

$$\frac{dy}{dx}.\frac{dz}{dy} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial \phi}{\partial y}\right)/\left(\frac{\partial f}{\partial y}\right).\left(\frac{\partial \phi}{\partial z}\right)$$

or $\left(\frac{\partial f}{\partial y}\right).\left(\frac{\partial \phi}{\partial z}\right)\frac{dz}{dx} = \left(\frac{\partial f}{\partial x}\right)/\left(\frac{\partial \phi}{\partial y}\right)$. Hence Proved

**Example 16 :** If the curves $f(x,y)=0$ and $\phi(x, y)=0$ touch, show that at the point of contact $\frac{\partial f}{\partial x}.\frac{\partial \phi}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial \phi}{\partial x} = 0$

**Solution :** For the curve $f(x, y)=0$, we have

$\frac{dy}{dx} = -\left(\frac{\partial f}{\partial x}\right)/\left(\frac{\partial f}{\partial y}\right)$ and for the curve $\phi(x, y)=0$, $\frac{dy}{dx} = -\left(\frac{\partial \phi}{\partial x}\right)/\left(\frac{\partial \phi}{\partial y}\right)$

Also if two curves touch each other at a point then at that point the values of (dy/dx) for the two curves must be the same,

Hence at the point of contact

$$-\left(\frac{\partial f}{\partial x}\right)/\left(\frac{\partial f}{\partial y}\right) = -\left(\frac{\partial \phi}{\partial x}\right)/\left(\frac{\partial \phi}{\partial y}\right)$$

or $\left(\frac{\partial f}{\partial x}\right).\left(\frac{\partial \phi}{\partial y}\right)-\left(\frac{\partial \phi}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)=0.$ Hence Proved

**Example 17 :** If $\phi(x,y,z)=0$ show that

$$\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y\left(\frac{\partial x}{\partial y}\right)_z=-1$$

(U.P.T.U. 2004)

**Solution :** The given relation defines y as a function of x and z. treating x as constant

$$\left(\frac{\partial y}{\partial z}\right)_x=-\frac{\partial\phi/\partial z}{\partial\phi/\partial y}\text{................(i)}$$

The given relation defines z as a function of x and y. Treating y as constant

$$\left(\frac{\partial z}{\partial x}\right)_y=-\frac{\partial\phi/\partial x}{\partial\phi/\partial z}\text{.........(ii)}$$

Similarly, $\left(\frac{\partial x}{\partial z}\right)_z=-\frac{\partial\phi/\partial y}{\partial\phi/\partial x}$.............(iii)

Multiplying (i), (ii) and (iii) we get

$$\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y\left(\frac{\partial x}{\partial y}\right)_z=-1$$ Hence Proved.

**Change of Variables :** If u is a function of x, y and x, y are functions of t and r, then u is called a composite function of t and r.

Let u =f(x, y) and x = g(t, r), y = h(t, r) then the continuous first order partial derivatives are

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}.\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}.\frac{\partial y}{\partial t}$$

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}.\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}.\frac{\partial y}{\partial r}$$

**Example 18 :** If $u=u\left(\frac{y-x}{xy},\frac{z-x}{xz}\right)$ show that $x^2\frac{\partial u}{\partial x}+y^2\frac{\partial u}{\partial y}+z^2\frac{\partial u}{\partial z}=0$

(U.P.T.U 2005)

**Solution :** Here given $u=u\left(\frac{y-x}{xy},\frac{z-x}{xz}\right)$

$=u(r,s)$

where $r=\frac{y-x}{xy}$ and $s=\frac{z-x}{zx}$

$$\Rightarrow r=\frac{1}{x}-\frac{1}{y} \text{ and } s=\frac{1}{x}-\frac{1}{z} \text{ ...............(i)}$$

we know that

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}.\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}.\frac{\partial s}{\partial x}$$

$$=\frac{\partial u}{\partial r}\left(-\frac{1}{x^2}\right)+\frac{\partial u}{\partial s}\left(-\frac{1}{x^2}\right) \qquad \because r=\frac{1}{x}-\frac{1}{y}$$

$$\Rightarrow \frac{\partial r}{\partial x}=-\frac{1}{x^2}$$

$$=-\frac{1}{x^2}\frac{\partial u}{\partial r}-\frac{1}{x^2}\frac{\partial u}{\partial s} \qquad \because s=\frac{1}{x}-\frac{1}{z}$$

$$\Rightarrow \frac{\partial s}{\partial x}=-\frac{1}{x^2}$$

$$\text{or } x^2\frac{\partial u}{\partial x}=-\frac{\partial u}{\partial r}-\frac{\partial u}{\partial s} \text{ ..................(ii)}$$

$$\text{Similarly } \frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}.\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}.\frac{\partial s}{\partial y}$$

$$=\frac{\partial u}{\partial y}.\frac{1}{y^2}+\frac{\partial u}{\partial s}.0 \qquad \text{from (i)}$$

$$\text{or } y^2\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r} \text{ ...................(iii)}$$

$$\text{and } \frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}.\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}.\frac{\partial s}{\partial z}$$

$$=\frac{\partial u}{\partial r}.0+\frac{\partial u}{\partial s}.\frac{1}{z^2}$$

$$\Rightarrow z^2\frac{\partial u}{\partial z}=\frac{\partial u}{\partial s} \text{ ......................(iv)}$$

Adding (i) (ii) and (iii) we get

$$x^2\frac{\partial u}{\partial x}+y^2\frac{\partial u}{\partial y}+z^2\frac{\partial u}{\partial z}=0 \qquad \text{Hence Proved.}$$

**Example 19 :** If u = u (y - z, z - x, x - y) Prove that

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$$

**Solution :** Here given u = u(y - z, z - x, x - y)

Let X = y - z, Y = z - x and Z = x - y...............(i)

Then u = u (X,Y,Z), where X, Y, Z are function of x,y and z.

Then

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial X}\frac{\partial X}{\partial x}+\frac{\partial u}{\partial Y}\frac{\partial Y}{\partial x}+\frac{\partial u}{\partial Z}\frac{\partial Z}{\partial x}\text{ .................(ii)}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial X}\frac{\partial X}{\partial y}+\frac{\partial u}{\partial Y}\frac{\partial Y}{\partial y}+\frac{\partial u}{\partial Z}\frac{\partial Z}{\partial y}\text{ ................(iii)}$$

and $\frac{\partial u}{\partial z}=\frac{\partial u}{\partial X}\frac{\partial X}{\partial z}+\frac{\partial u}{\partial Y}\frac{\partial Y}{\partial z}+\frac{\partial u}{\partial Z}\frac{\partial Z}{\partial z}$ ...........(iv)

with the help of (i), equations (ii), (iii) and (iv) gives.

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial X}0+\frac{\partial u}{\partial Y}(-1)+\frac{\partial u}{\partial Z}(1)=-\frac{\partial u}{\partial Y}+\frac{\partial u}{\partial Z}\text{ ...................(v)}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial X}.1+\frac{\partial u}{\partial Y}.0+\frac{\partial u}{\partial Z}(-1)=\frac{\partial u}{\partial X}-\frac{\partial u}{\partial Z}\text{ ....................(vi)}$$

and $\frac{\partial u}{\partial z}=\frac{\partial u}{\partial X}(-1)+\frac{\partial u}{\partial Y}(1)+\frac{\partial u}{\partial Z}(0)=-\frac{\partial u}{\partial X}+\frac{\partial u}{\partial Y}$ ..............(vii)

Adding (v), (vi) and (vii) we get $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0.$ Hence Proved.

**Example 20 :** If z is a function of x and y and $x = e^u + e^{-v}$, $y = e^{-u} - e^v$

Prove that $\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}$

(AV.UP 2005)

**Solution :** Here z is a function of x and y, where x and y are functions of u and v.

$$\therefore\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\text{ ...............(i)}$$

and $\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}.\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$ ..........(ii)

Also given that

$x = e^u + e^{-v}$ and $y = e^{-u} - e^v$

$$\therefore\frac{\partial x}{\partial u}=e^u,\ \frac{\partial x}{\partial v}=-e^{-v},\ \frac{\partial y}{\partial u}=-e^{-u},\ \frac{\partial y}{\partial v}=-e^v$$

$\therefore$ From (i) we get

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}(e^u)+\frac{\partial z}{\partial y}(-e^{-u})\text{ ...............(iii)}$$

and from (ii) we get

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\left(-e^{-v}\right)+\frac{\partial z}{\partial y}\left(-e^{-v}\right)\text{ ............(iv)}$$

Subtracting (iv) from (iii) we get

$$\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=\left(e^{u}+e^{-v}\right)\frac{\partial z}{\partial x}-\left(e^{-u}-e^{v}\right)\frac{\partial z}{\partial y}$$

$$=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}. \qquad \text{Hence Proved.}$$

**Example 21 :** If $V = f(2x -3y, 3y -4z, 4z -2x)$, compute the value of $6V_x +4V_y +3V_z$.

(U.P.T.U. 2009)

**Solution :** Here given $V = f(2x -3y, 3y -4z, 4z -2x)$
Let $X = 2x-3y$, $Y =3y-4z$ and $Z = 4z-2x$............(i)
Then $u = f(X, Y, Z)$, where X, Y, Z are function of x, y and z.

$$\text{Then } V_x=\frac{\partial V}{\partial x}=\frac{\partial V}{\partial X}\frac{\partial X}{\partial x}+\frac{\partial V}{\partial Y}\frac{\partial Y}{\partial x}+\frac{\partial V}{\partial Z}\frac{\partial Z}{\partial x}\ ................(ii)$$

$$V_y=\frac{\partial V}{\partial y}=\frac{\partial V}{\partial X}\frac{\partial X}{\partial y}+\frac{\partial V}{\partial Y}\frac{\partial Y}{\partial y}+\frac{\partial V}{\partial Z}\frac{\partial Z}{\partial y}\ ......................(iii)$$

$$\text{and } V_z=\frac{\partial V}{\partial z}=\frac{\partial V}{\partial X}\frac{\partial X}{\partial z}+\frac{\partial V}{\partial Y}\frac{\partial Y}{\partial z}+\frac{\partial V}{\partial Z}\frac{\partial Z}{\partial z}\ .................(iv)$$

with the help of (i), equations (ii), (iii) and (iv) gives

$$V_x=\frac{\partial V}{\partial X}(2)+\frac{\partial V}{\partial Y}(0)+\frac{\partial V}{\partial Z}(-2)$$

$$\text{or } V_x=2\left(\frac{\partial V}{\partial X}-\frac{\partial V}{\partial Z}\right)$$

$$\Rightarrow 6V_x=12\left(\frac{\partial V}{\partial X}-\frac{\partial V}{\partial Z}\right)\ ..................(v)$$

$$\text{Now } V_y=\frac{\partial V}{\partial X}(-3)+\frac{\partial V}{\partial Y}(3)+\frac{\partial V}{\partial Z}(0)$$

$$\text{or } V_y=3\left(-\frac{\partial V}{\partial X}+\frac{\partial V}{\partial Y}\right)$$

$$\Rightarrow 4V_y=12\left(-\frac{\partial V}{\partial X}+\frac{\partial V}{\partial Y}\right)\ ..................(vi)$$

$$\text{and } V_z=\frac{\partial V}{\partial X}(0)+\frac{\partial V}{\partial Y}(-4)+\frac{\partial V}{\partial Z}(4)$$

$$\text{or } V_z=4\left(-\frac{\partial V}{\partial Y}+\frac{\partial V}{\partial Z}\right)$$

$$\Rightarrow 3V_z=12\left(-\frac{\partial V}{\partial Y}+\frac{\partial V}{\partial Z}\right)\ ................(vii)$$

Adding (v), (vi) and (vii) we get
$6V_x + 4V_y +3V_z =0$ Answer.

**Example 22 :** If x +y = 2e$^\theta$ cos$\phi$ and x -y = 2 ie$^\theta$ sin$\phi$, Prove that

$$\frac{\partial^2 V}{\partial\theta^2}+\frac{\partial^2 V}{\partial\phi^2}=4xy\frac{\partial^2 V}{\partial x\partial y}$$

(U.P.T.U. 2001; U.P.P.C.S. 1997)

**Solution :** we have x+y = 2e$^\theta$ cos$\phi$, x-y =2e$^\theta$ i sin$\phi$

$\Rightarrow$ x = e$^\theta$ (cos$\phi$ + sin$\phi$) = e$^\theta$. e$^{i\phi}$

or x = e$^{\theta + i\phi}$

and y = e$^\theta$(cos$\phi$ - i sin$\phi$) = e$^\theta$. e$^{-i\phi}$

or y = e$^{\theta - i\phi}$

since we know

$$\frac{\partial V}{\partial\theta}=\frac{\partial V}{\partial x}\frac{\partial x}{\partial\theta}+\frac{\partial V}{\partial y}\frac{\partial y}{\partial\theta}$$

$$=\frac{\partial V}{\partial x}.e^{\theta+i\phi}+\frac{\partial V}{\partial y}.e^{\theta-i\phi}$$

$$=x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}$$

$$\Rightarrow\frac{\partial}{\partial\theta}\equiv x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\text{ ................(i)}$$

$$\text{and }\frac{\partial V}{\partial\phi}=\frac{\partial V}{\partial x}\frac{\partial x}{\partial\phi}+\frac{\partial V}{\partial y}\frac{\partial y}{\partial\phi}$$

$$=\frac{\partial V}{\partial x}.ie^{\theta+i\phi}+\frac{\partial V}{\partial y}(-1).ie^{\theta-i\phi}$$

$$=ix\frac{\partial V}{\partial x}-iy\frac{\partial V}{\partial y}$$

$$\Rightarrow\frac{\partial}{\partial\phi}=ix\frac{\partial}{\partial x}-iy\frac{\partial}{\partial y}\text{ ...............(ii)}$$

$$\text{Again }\frac{\partial^2 V}{\partial\theta^2}+\frac{\partial^2 V}{\partial\phi^2}=\left(\frac{\partial}{\partial\theta}+i\frac{\partial}{\partial\phi}\right)\left(\frac{\partial V}{\partial\theta}-i\frac{\partial V}{\partial\phi}\right)$$

$$=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}-x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\times\left(x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+x\frac{\partial V}{\partial x}-y\frac{\partial V}{\partial y}\right)$$

using (i) and (ii)

$$=2y\frac{\partial}{\partial y}\left(2x\frac{\partial V}{\partial x}\right)$$

$$=4y\left[x\frac{\partial^2 V}{\partial y\partial x}+\frac{\partial V}{\partial x}.0\right]$$

$= 4xy\dfrac{\partial^2 V}{\partial x \partial y}$. Hence Proved.

**Example 23 :** Transform the equation $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ into polar Co-ordinates

(A.U.U.P. 2003, Delhi college of Engg. 2004, G.G.S.I.P.U. 2004)

**Solution :** The relation Connecting Cartesian Co-ordinates (x, y) with polar Co-ordinates (r, θ) are

$x = r\cos\theta,\ y = r\sin\theta$

Squaring and adding $r^2 = x^2 + y^2$

Dividing $\tan\theta = \dfrac{y}{x}$

$\therefore r = \sqrt{x^2+y^2},\ \ \theta = \tan^{-1}\left(\dfrac{y}{x}\right)$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta, \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{r\sin\theta}{r} = \sin\theta$$

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r}$$

and $\dfrac{\partial\theta}{\partial y} = \dfrac{1}{1+\frac{y^2}{x^2}}.\dfrac{1}{x} = \dfrac{x}{x^2+y^2} = \dfrac{\cos\theta}{r}$

Here u is a composite function of x and y

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\Rightarrow \frac{\partial}{\partial x} \equiv \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Also $\dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial r}.\dfrac{\partial r}{\partial y} + \dfrac{\partial u}{\partial\theta}.\dfrac{\partial\theta}{\partial y}$

$$= \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}$$

$$\Rightarrow \frac{\partial}{\partial y} \equiv \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta} \ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Now we shall make use of the equivalence of Cartesian & polar operators as given by (i) and (ii)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$= \cos\theta\left[\cos\theta\frac{\partial^2 u}{\partial r^2} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\left(-\frac{1}{r^2}\right) - \frac{\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta}\right] - \frac{\sin\theta}{r}\left[-\sin\theta\frac{\partial u}{\partial r} + \cos\theta\frac{\partial^2 u}{\partial\theta\partial r} - \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\theta}{r}\frac{\partial^2 u}{\partial\theta^2}\right]$$

$$= \cos^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{2\cos\theta\sin\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} - \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} \text{......................(iii)}$$

And $\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)$

$$= \sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$= \sin\theta\left[\sin\theta\frac{\partial^2 u}{\partial r^2} - \frac{\cos\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta}\right] + \frac{\cos\theta}{r}\left[\cos\theta\frac{\partial u}{\partial r} + \sin\theta\frac{\partial^2 u}{\partial\theta\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta} + \frac{\cos\theta}{r}\frac{\partial^2 u}{\partial\theta^2}\right]$$

$$= \sin^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{2\cos\theta\sin\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} + \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial^2 u}{\partial\theta^2} \text{...........(iv)}$$

Adding (iii) and (iv)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

Therefore $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ transforms into $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0$

**Exercise :**

1. If $u = \tan^{-1}\frac{xy}{\sqrt{1+x^2+y^2}}$, show that

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{1}{\left(1+x^2+y^2\right)^{3/2}}$$

2. If $u(x+y) = x^2 + y^2$, prove that

$$\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)$$

3. If $u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$, prove that

$$\frac{\partial^2 u}{\partial y\partial x} = \frac{x^2 - y^2}{x^2 + y^2}$$

(U.P.T.U. Special Exam 2001)

4. If z = f(x+ay) +ϕ(x −ay), prove that

$$\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

5. If $u = 2(ax+by)^2 - (x^2+y^2)$ and $a^2+b^2=1$, find the value of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$.

Ans = 0

6. If $V = (x^2+y^2+z^2)^{-1/2}$, Show that

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

7. If $u = (x^2+y^2+z^2)^{1/2}$, then prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2}{u}$$

8. If $u = x^y$, show that $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$

(P.T.U. 1999)

9. If $\theta = t^n e^{-\frac{r^2}{4t}}$, find the value of n which will make $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$

Ans. $n = -\frac{3}{2}$

10. If x = r cosθ, y = r sinθ, prove that

(i) $\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$ (ii) $\frac{1}{r}\frac{\partial x}{\partial \theta} = r\frac{\partial \theta}{\partial x}$ (iii) $\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$

11. If $z = e^{ax+by} f(ax - by)$, Show that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$

12. If V = f(r) and $r^2 = x^2+y^2+z^2$, Prove that $V_{xx}+V_{yy}+V_{zz} = f''(r) + \frac{2}{r} f'(r)$

13. Find p and q, if $x = \sqrt{a}$ (sin u + cos v), $y = \sqrt{a}$ (cos u - sin v), z = 1+sin(u - v) where p and q means $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ respectively.

[ **Hint.** $x^2+y^2 = 2az$ $\therefore z = \frac{x^2+y^2}{2a}$ ]

Ans. $p = \frac{x}{a}$, $q = \frac{y}{a}$

14. If u =(1-2xy +y²)$^{-1/2}$, Prove that $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = y^2u^3$

(M.D.U. 2001)

15. Verify Euler's theorem for the function $z = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$

(Delhi college of Engg., March 2004)

16. If $u = \sin^{-1}\frac{x^2+y^2}{x+y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$

(B.P.S.C. 1997; U.P.P.C.S. 1990; J.N.T.U. 1999, G.G.S.I.P.U. 2007, M.D.U. 2002, 03 V.T.U. 2003)

17. If $z = \tan^{-1}\frac{x^3+y^3}{x-y}$, Prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \sin 2z$

(U.P.T.U. 2001, M.D.U. 2002, G.G.S.I.P.U. 2006)

18. If $u = \log\left(\frac{x^4+y^4}{x+y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$

(U.P.T.U. 2001)

19. If $u = \cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$

(G.G.S.I.P.U. 2006, V.T.U. 2004)

20. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2\cos 3u \sin u$

(U.P.P.C.S. 1993)

(A.U.U.P. 2005, P.T.U. 2002, Delhi college of Engg 2004.)

21. If $u = \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, Prove that

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\tan u$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin u \cos 2u}{4\cos^3 u}$

(A.U.U.P. 2005, M.D.U. 2000, 2004, Delhi college of Engg. 2005)

22. If $u = \sin^{-1}\left\{\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right\}^{1/2}$, then show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}(13 + \tan^2 u)$$

(Delhi college of Engg. 2004, M.D.U. 2001, 2003)

23. If $f(x,y)= \frac{1}{x^2}+\frac{1}{xy}+\frac{\log x-\log y}{x^2+y^2}$, show that $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+2f(x,y)=0$

(PTU 2004)

24. If $z = x^4 y^2 \sin^{-1}\left(\frac{x}{y}\right)$ + log x - log y show that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=6x^4y^2\sin^{-1}\left(\frac{x}{y}\right)$

(U.P.T.U. 2003)

25. Show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=2\tan u$ where $u = \sin^{-1}\frac{x^3+y^3+z^3}{ax+by+cz}$

(U.P.T.U. 2003)

26. If $V = \log_e \sin\left\{\frac{\pi\left(2x^2+y^2+xz\right)^{1/2}}{2\left(x^2+xy+2yz+z^2\right)^{1/3}}\right\}$ Prove that when x =0, y =1, z =2

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}=\frac{\pi}{12}$$

(U.P.P.C.S. 1991)

27. If $u = x^2 - y^2 + \sin yz$ where $y = e^x$ and $z = \log x$; find $\frac{du}{dx}$

Ans. $2(x - e^{2x})+e^x \cos(e^x \log x)\left(\log x + \frac{1}{x}\right)$.

28. If u = f(r, s) and r = x+y, s = x-y; show that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=2\frac{\partial u}{\partial r}$

29. If $x = e^r\cos\theta$, $y = e^r \sin\theta$, show that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=e^{-2r}\left(\frac{\partial^2 u}{\partial r^2}+\frac{\partial^2 u}{\partial \theta^2}\right)$

30. If z is a function of x and y and u and v be two other variables such that $u = lx + my$, $v = ly - mx$, show that $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\left(l^2+m^2\right)\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}\right)$

31. If z = f(x, y), where x = uv and $y = \frac{u}{v}$, then show that $\frac{\partial z}{\partial y}=\frac{v}{2}.\frac{\partial z}{\partial u}-\frac{v^2}{2u}.\frac{\partial z}{\partial v}$

(A.U.U.P. 2005)

32. If $z = \sqrt{x^2+y^2}$ and $x^3 +y^3 +3axy = 5a^2$, find the value of $\frac{dz}{dx}$ when x = y =a.

33. If u = f(y/x) show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$

(U.P.P.C.S. 1997)

34. Prove that if $z = \phi(y + ax) + \psi(y - ax)$ then $a^2\frac{\partial z^2}{\partial y^2} - \frac{\partial^2 z}{\partial x^2} = 0$ for any twice differentiable $\phi$ and $\psi$, a is a constant.

(I.A.S. 2007)

35. If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ Prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$ (I.A.S. 1996)

**Tick the Correct answer from the choices given below**

1. If $u = f(y/x)$, then

(i) $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0$ (ii) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

(iii) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -u$ (iv) $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 1$

Ans. (ii)

(I.A.S. 1999)

2. If $z = f(x+ay) + \phi(x-ay)$ then

(i) $\frac{\partial^2 z}{\partial x^2} = a^2\frac{\partial^2 z}{\partial y^2}$ (ii) $\frac{\partial^2 z}{\partial y^2} = -\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2}$

(iii) $\frac{\partial^2 z}{\partial y^2} = a^2\frac{\partial^2 z}{\partial x^2}$ (iv) $\frac{\partial^2 z}{\partial x^2} = 2a^2\frac{\partial^2 z}{\partial y^2}$

Ans. (iii)

(I.A.S. 1999)

3. If $u = e^{xyz}$, then $\frac{\partial^3 u}{\partial x\partial y\partial z}$ is equal to

(i) $(x^2+y^2+z^2+3xyz)e^{xyz}$ (ii) $(1+x^2y^2z^2+3xyz)e^{xyz}$

(iii) $(1+x^2+y^2+z^2)e^{xyz}$ (iv) $(3x+3y+3z+x^2y^2z^2)e^{xyz}$

Ans. (ii)

4. If $u = \log_e(x^3+y^3+z^3-3xyz)$, then $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$ is equal to

(i) $\frac{1}{2}(x+y+z)$ (ii) $\frac{1}{3}(x+y+z)$

(iii) $\frac{2}{x+y+z}$ (iv) $\frac{3}{x+y+z}$

Ans. (iv)

5. If $z = xy\,f(y/x)$, then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$ is equal to

(i) z (ii) 2z

(iii) xz (iv)yz

Ans. (ii)
(U.P.P.C.S. 1994)

6. $\text{Sin}^{-1}\left(\frac{y}{x}\right)$ is a homogeneous function of x, y of degree

(i) 1 (ii) 2
(iii) 3 (iv) 0

Ans. (iv)
(U.P.P.C.S. 1994, 1995)

7. If z = uv
$u^2 + v^2 - x - y = 0$
$u^2 - v^2 + 3x + y = 0$
Then $\frac{\partial z}{\partial x}$ is equal to

(i) u + v (ii) $\frac{2u^2 - v^2}{2uv}$
(iii) $\frac{3u^2 + v^2}{2uv}$ (iv) $\frac{u^2 - 3v^2}{2uv}$

Ans. (ii)
(I.A.S. 1994)

8. If $u = \sin^{-1}\left(\frac{x^2 + y^2}{x + y}\right)$, then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$ is

(i) f (ii) 2f
(iii) tan f (iv) sin f

Ans. (iii)
(I.A.S. 1994, U.P.P.C.S. 1994)

9. If $u = x^4 y^2$, where $x = t^2$ and $y = t^3$, then $\frac{du}{dt}$ is

(i) $22t^{13}$ (ii) $14\ t^{22}$
(iii) $\frac{22}{23}t^{11}$ (iv) $14t^{13}$

Ans. (iv)
(R.A.S 1993)

10. If $u = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)$ the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ is

(i) tan 2u (ii) cos 2u
(iii) sin 2u (iv) $\sec^2 2u$

Ans. (iii)

(R.A.S. 1994)

11. If $x^x y^y z^z = c$, then $\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-n}$ when $(x = y = z)$ if n is equal to

(i) $\frac{1}{2}$ (ii) $\frac{1}{3}$

(iii) 1 (iv) 2

Ans. (iii)

(U.P.P.C.S. 1994)

12. If $u = x \log xy$, where $x^3 + y^3 + 3xy = 1$, then $\frac{du}{dx}$ is equal to

(i) $(1 + \log xy) - \frac{x}{y}\left(\frac{x^2 + y}{y^2 + x}\right)$ (ii) $(1 + \log xy) - \frac{y}{x}\left(\frac{y^2 + x}{x^2 + y}\right)$

(iii) $(1 - \log xy) - \frac{x}{y}\left(\frac{x^2 + y}{y^2 + x}\right)$ (iv) $(1 - \log xy) - \frac{y}{x}\left(\frac{y^2 + x}{x^2 + y}\right)$

Ans. (i)

13. If $u = \log\left(\frac{x^2 + y^2}{x + y}\right)$ Then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ is equal to

(i) 0 (ii) 1

(iii) u (iv) eu

Ans. (ii)

14. If $u = \sin^{-1}\frac{x^{1/2} - y^{1/2}}{x^{1/2} + y^{1/2}}$, then $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$ is equal to

(i) 0 (ii) $-\frac{u}{4}$

(iii) $\frac{u}{4}$ (iv) $\frac{3u}{4}$

Ans. (i)

15. If $u = \sqrt{x^2 + y^2 + z^2}$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$ is equal to

(i) 0 (ii) u

(iii) $\frac{1}{2u}$ (iv) $\frac{u}{2}$

Ans. (ii)

16. If $u = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$, then $u_x + u_y + u_z$ is equal to

(i) 0 (ii) 1

(iii) $x + y + z$ (iv) $2(x + y + z)$

Ans. (i)

17. If $u = f(y-z, z-x, x-y)$ then $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$ is equal to

(i) $x + y + z$ (ii) $1 + x + y + z$

(iii) 1 (iv) 0

Ans. (iv)
(U.P.P.C.S. 1994)

18. If $z = f(r)$ and $r^2 = x^2 + y^2$, then $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$ is equal to

(i) $f''(r) + r f'(r)$ (ii) $f'(r) + r f''(r)$

(iii) $f'(r) + \frac{1}{r} f''(r)$ (iv) $f''(r) + \frac{1}{r} f'(r)$

Ans. (iv)

19. If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, then $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2$ is equal to

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$ (ii) $\frac{1}{2}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right)$

(iii) $2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right)$ (iv) None of these

Ans. (iii)

20 If $u = r^m$ where $r^2 = x^2 + y^2 + z^2$, then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ is equal to

(i) $r^{m-2}$ (ii) $m(m-1)r^{m-2}$

(iii) $m(m+1)r^{m-2}$ (iv) $(m^2-1)r^{m-2}$

Ans. (iii)

21. If u is a homogeneous function of degree n in x, y, z and if $u = f(x, y, z)$ where x, y, z are the first derivatives of u with respect to x, y, z respectively then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}$ is equal to

(i) nu (ii) $n(n-1)u$

(iii) $\frac{1}{n-1}u$ (iv) $\frac{n}{n-1}u$

Ans. (i)

22. The total differential dz of z which is a function of x and y is equal to

(i) $dz = dx + dy$ (ii) $dz = \frac{dz}{dx}dx + \frac{dz}{dy}dy$

(iii) $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$ (iv) $\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$

Ans. (iii)

23. If $f(x, y) = 0$, $\phi(y, z) = 0$, then

(i) $\frac{\partial f}{\partial y}.\frac{\partial \phi}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial \phi}{\partial y}.\frac{dz}{dx}$ (ii) $\frac{\partial f}{\partial y}.\frac{\partial \phi}{\partial z}.\frac{dz}{dx} = \frac{\partial f}{\partial x}.\frac{\partial \phi}{\partial x}$

(iii) $\frac{\partial f}{\partial y}.\frac{\partial \phi}{\partial z}.\frac{dz}{dx} = \frac{\partial f}{\partial x}.\frac{\partial \phi}{\partial y}$ (iv) None of these

Ans. (iii)

24. If $u = y^x$, then $\frac{\partial u}{\partial x}$ is equal to

(i) $y^x(\frac{x}{y} + \log y)$ (ii) $xy^{x-1}$

(iii) $y^x \log y$ (iv) $y^x \log x$

Ans. (iii)

25. A function f(x, y) is said to be homogenous of degree n in x, y if

(i) $f(tx, ty) = t^{2n} f(x, y)$

(ii) $f(tx, ty) = t^{n-1} f(x, y)$

(iii) t is of the form $x^n f(x/y)$

(iv) t is of the form $x^n f(y/x)$

Ans. (iv)

26. Consider the Assertion (A) and Reason (R) given below :

Assertion (A) - If $u = xy\, f(y/x)$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$

Reason (R) - Given function u is homogenous of degree 2 in x and y.
of these statements -

(i) Both A and R are true and R is the correct explanation of A.

(ii) Both A and R are true and R is not the correct explanation of A.

(iii) A is true but R is false.

(iv) A is false but R is true.

Ans. (i)

27. Consider the following statements :

Assertion (A) - for $x = r\cos\theta$, $y = r\sin\theta$ $\frac{\partial x}{\partial r} = \frac{1}{\partial r / \partial x}$ and $\frac{\partial x}{\partial \theta} = \frac{1}{\partial \theta / \partial x}$

Reason (R) $\frac{dy}{dx} = \frac{1}{dx / dy}$

of these statement -
(i) Both A and R are true and R is the correct explanation of A.
(ii) Both A and R are true and R is not the correct explanation of A.
(iii) A is true but R is false.
(iv) A is false but R is true.

Ans. (iv)
(I.A.S. 1995)

28. If $x = r\cos\theta$, $y = r\sin\theta$, then $\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix}$ is (U.P.P.C.S. 1994)

(i) $r$ (ii) $\frac{1}{r}$

(iii) $2r$ (iv) $\frac{1}{2r}$

Ans. (i)

29. Match the list I with II
List I

(a) If $u = \frac{x^2 y}{x+y}$, then $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y}$

(b) If $u = \frac{x^{1/2} - y^{1/2}}{x^{1/4} + y^{1/4}}$, then $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$

(c) If $u = x^{1/2} + y^{1/2}$ then $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$

(d) If $u = f(y/x)$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

List II

(p) $-\frac{3}{16}u$

(q) $2\frac{\partial u}{\partial x}$

(r) 0

(s) $-u/4$

correct match is

| | a | b | c | d |
|---|---|---|---|---|
| (i) | p | q | r | s |
| (ii) | q | p | s | r |
| (iii) | q | p | r | s |
| (iv) | p | q | s | r |

Ans. (ii)

29. If $u = 3x^2yz + 2yz^3 + 6\,x^4$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$ equal to

(U.P.P.C.S. 1995)

(i) $6x^2\,yz + 4xy^3 + 12\,x^4$
(ii) $3x^2yz + 2yz^3 + 6x^4$
(iii) $12\,x^2yz + 6yz^3 + 12\,x^4$
(iv) $12\,x^2yz + 8yz^3 + 24\,x^4$

Ans. (iv)

30. If $f(x, y) = x^4+x^2y^2+y^4$, then $\frac{\partial^2 f}{\partial x \partial y}$ is equal to

(U.P.P.C.S 1994)

(i) $4xy$ (ii) $12\,x^2 + 2y^2$
(iii) $2x^2 + 12\,y^2$ (iv) $4x^3 + 2xy^2$

Ans. (i)

# Chapter 3

# Curve Tracing

## Rules for tracing Cartesian Curves :-

**I. Symmetry** - The most important point while tracing the curves is to judge its symmetry which we do as following:

(a) If the equation of the curve involves even and only even powers of x, then there is symmetry about y axis. The reason is that if we obtain a certain value of y by putting x =a in the equation then the same value of y will be obtained by putting x = -a in the equation because it contains even and only even powers in x e.g. the parabola $x^2 = 4ay$ and the circle $x^2+y^2=a^2$ are both symmetrical about y axis.

**Note.** The stress on the words 'even and only even' should be observed. $x^2+y^2 = ax$ is not symmetrical about y axis as it involves odd powers of x as well.

(b) Similarly, if the equation of the curve involves even and only even powers of y then the curve is symmetrical about x axis, e.g. ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and parabola $y^2 = 4ax$ are symmetrical about x axis.

(c) If the equation of the curve involves even and only even powers of x as well as of y, then the curve is symmetrical about both the axes i.e. the circle $x^2+y^2=a^2$ or ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are both symmetrical about the axes.

(d) If x be changed into -x and y be changed into -y, and the equation of the carve remains unchanged, then the there is symmetry in opposite quadrants e.g. $xy=a^2$ is the equation of the hyperbola referred to asympototes as axes which is symmetrical in 1st and 3rd quadrants.

(e) If x be changed into y and y into x and the equation of the curve remains unchanged, then curve is symmetrical about the line y = x i.e. a line passing through origin and making an angle $45^0$ with positive direction of x-axis.

**Note.** In this case solve the equation of the curve with the line y =x and find the point of intersection e.g. $x^3+y^3=3axy$ is symmetrical about y =x and cuts it in points (0, 0) and $\left(\frac{3a}{2},\frac{3a}{2}\right)$.

**II Points**

We should find points where the curve cuts the axes which are obtained as follows;

Put y = 0 in the equation of the curve and solve for x and thus you will find the points where the curve cuts the x axis e.g. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ putting y =0 we get $x^2 = a^2$.

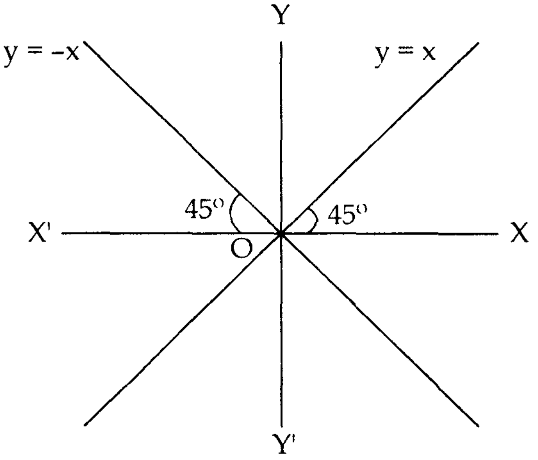
$\therefore$ x =a and -a, and hence ellipse cuts the x axis in points (a, 0) and (-a, 0) Similarly, put x =0 in the equation of the curve and solve for y and thus you will find the points where the curve cuts the y axis e.g. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ putting x =0, we get y =b and -b and hence the ellipse cuts the y axis in points (0, b) and (0, -b).

Again $y^2 = 4ax$ putting x =0, we get y =0 and it we put y =0 we get x =0 and hence the point (0, 0) i.e. origin lies on the curve. The best way for detecting whether the origin lies on the curve is to see that the equation does not contain any constant term just as $y^2 = 4ax$ contains no constant term and hence it passes through the origin. The circle $x^2 + y^2 = a^2$ contains the constant term $a^2$ and hence it does not pass through the origin.

**III. Tangents :** After we have found the points which lies on the curve we shall try to find tangents at those points.

**(a) Tangents at origin :** In case we find that origin is a point on the curve, then the tangents at the origin are obtained by equating to zero the lowest degree terms occurring in the equation of the curve e.g. $y^2 = 4ax$ passes through origin and the lowest degree terms occurring in it is 4 ax which when equated to zero gives x =0 i.e. y axis which is tangent to the parabola at the origin.

Again the equation $a^2y^2 = a^2 x^2 - x^4$ contains no constant term and as such passes through the origin and the total lowest degree terms brought on one side are $a^2y^2 - a^2x^2$ which when equated to zero $y^2 - x^2 = 0$ or $y = \pm x$ which will be tangents at the origin to the curve i.e. the curve shall be touching both these two lines at the origin.



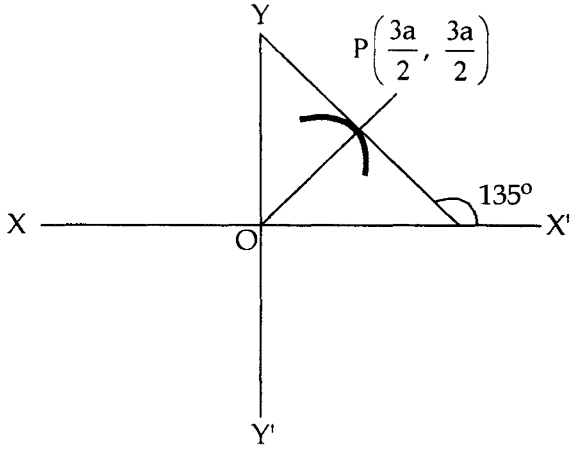
Again $x^3 + y^3 = 3axy$ passes through the origin and the lowest degree term is 3axy, which when equated to zero gives $x = 0$, $y = 0$ as the tangents at the origin i.e. curve will touch the axes at the origin.

Again $y^2(a+x) = x^2(3a-x)$ clearly passes through the origin and the total lowest degree terms brought on one side are $ay^2 - 3ax^2$ which when equated to zero gives $y^2 - 3x^2 = 0$ or $y = \pm\sqrt{3}\,x$ which will be tangents at the origin to the curve.

**Note.** We shall write tangents at the origin as $T_{0,\,0}$.

**(b) Tangents at any other points which is not origin :** Suppose there lines a point (h, k) on the curve, then in order to find the tangent at (h, k) we find the value of $\frac{dy}{dx}$ from the equation of the curve and substitute in it (h, k) which gives us the slope of the tangent at (h, k) and the equation of the tangent will be a line passing through (h, k) and with the above slope. If the slope comes out to be zero then the tangent will be parallel to x axis, e.g. we have seen before that $x^3 + y^3 = 3axy$ passes through $\left(\frac{3a}{2}, \frac{3a}{2}\right)$. Let us find $\frac{dy}{dx}$ by differentiating the equation of the curves w.r.t.x

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left(y + x\frac{dy}{dx}\right)$$



$\therefore \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$ and its value by substituting the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ is equal to -1 and hence the slope of the tangent at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ is -1 i.e., it will make an angle of 135°.

**IV. Asymptotes which are parallel to the axes**

The asymptotes parallel to the x axis may be obtained by equating to zero the coefficient of highest degree terms in x, provided it is not a constant.

Similarly, the asymptotes parallel to the axis of y may be obtained by equating to zero the coefficient of highest degree terms in y, provided it is not constant.

**V. Region**

If for some value of x greater than some quantity say a the corresponding values of y come out to be imaginary, then no part of the curve will lie beyond x =a. Similarly if for some value of y greater than some quantity b say the corresponding values of x come out to be imaginary, then no part of the curve shall lie beyond y =b. e.g. $y^2(2a - x) = x^3$

If x is greater than 2a, then 2a -x will be negative and hence $y^2$ will be negative. Therefore y will be imaginary for values of x greater than 2a and as such no part of this curve will lie beyond x =2a. Again if x is negative then 2a-x is positive but $x^3$ becomes negative and therefore $y^2$ will be negative and as such y imaginary. Hence no part of the curve shall exist in the negative side of x axis.

In case the equation of the curve can be arranged as a quadratic in y or x, then it is convenient to find the values of one variable in terms of the other and discuss the reality of the roots.

**Note.** The existence of loops is generally found by this.

**Example 1 :** Trace the curve $y^2(a - x) = x^3$ cissoid.

(U.P.T.U. 2005)

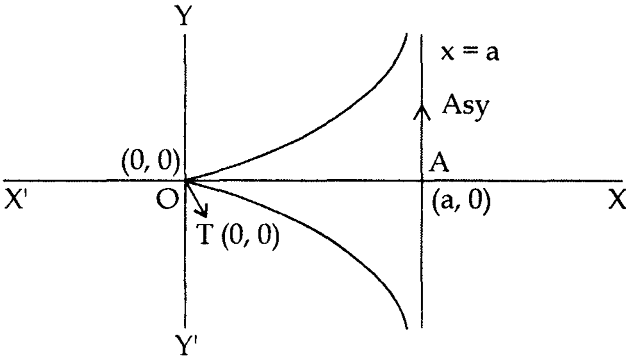
**Solution :** (i) Since there are even and only even powers of y, symmetry is about x-axis.

(ii) Putting $x = 0$ we get $y = 0$ and putting $y = 0$, we get $x = 0$ and hence (0, 0) is the only point on the curve.

(iii) Tangents at origin are given by the lowest degree terms equated to zero i.e. $ay^2 = 0 \therefore y = 0$ i.e. x-axis is tangent.

(iv) The equation is of 3rd degree and $x^3$ is present but is missing and hence equate to zero the coefficient of $y^2$ which is a-x. Therefore $x = a$ is an asymptote parallel to y axis.

(v) If either x is negative or greater than a, then corresponding values of y are imaginary. Therefore the curve is within the lines $x = 0$ and $x = a$ with the above five points the shape of the curve is as shown in the figure given below.



**Example 2 :** Trace the curve $y^2(a+x) = x^2(a-x)$ (Strophoid)

or

$(x^2 + y^2)\, x - a(x^2 - y^2) = 0$

(Uttarakhand TU 2006, U.P.P.C.S. 1996, B.P.S.C 2007)

**Solution :** (i) Symmetry about x -axis

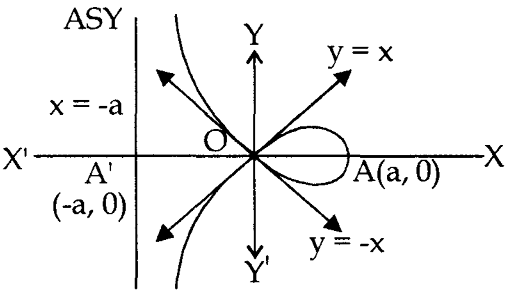
(ii) Points $y = 0$ then $x = 0$ and a, $\therefore$ (0, 0) and (a, 0) and when $x = 0$ then $y = 0 \therefore$ (0,0) Hence (0, 0) and (a, 0) are the only two points.

(iii) $T_{(0,\,0)}$ are $a(x^2 - y^2) = 0$ i.e. $y = \pm x$

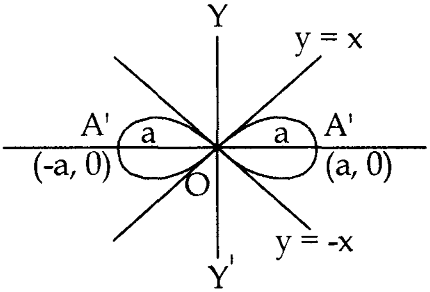
(iv) The asymptotes is $x + a = 0$ i.e. $x = -a$, on equating to zero the coefficient of $y^2$ as $y^3$ is missing.

(v) Again if x is greater than a then the value of $y^2$ is negative and hence y imaginary. Therefore the curve does not go beyond $x = a$, similarly it does not go beyond $x = -a$.

With the above five points the shape of the curve is as shown in the figure given below.



**Example 3:** Trace the curve $a^2 y^2 = x^2(a^2 - x^2)$
or $a^2 y^2 = a^2 x^2 - x^4$
**Solution :** (i) Symmetry about both axes.
(ii) Points (0, 0), (a, 0), (-a, 0)
(iii) $T_{0,0}$ are $y = \pm x$ and $T_{(a,0)}$ is y axis and $T_{(-a,0)}$ is y axis .
(iv) No asymptotes
(v) x cannot be greater than a in magnitude on either side of x axis with the above data the shape of the curve is as shown in figure.



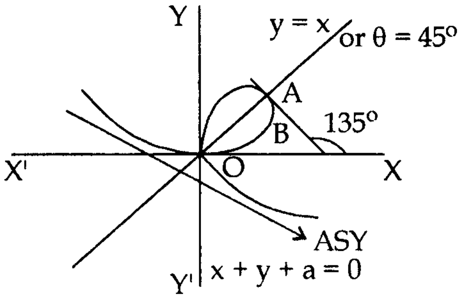
**Example 4 :** Trace the curve
$x^3 + y^3 = 3axy$ (Folium of Descartes)

(U.P.T.U. 2003)

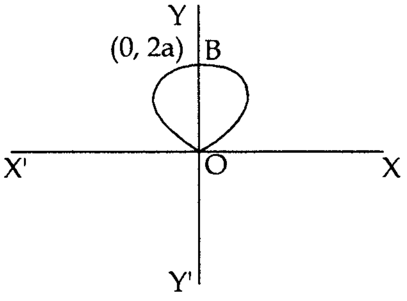
**Solution :** (i) If x be changed into y and y into x, the equation of the curve is unaltered and hence the symmetry is about the line y =x.
(ii) (0, 0) is one point and the other point is obtained by solving the equation with y =x which is $\left(\frac{3a}{2}, \frac{3a}{2}\right)$.

(iii) $T_{0,0}$ are x =0, y =0 and the value of $\frac{dy}{dt}$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ is -1, which shows that tangent at that point makes an angle of $135^0$ with x axis.

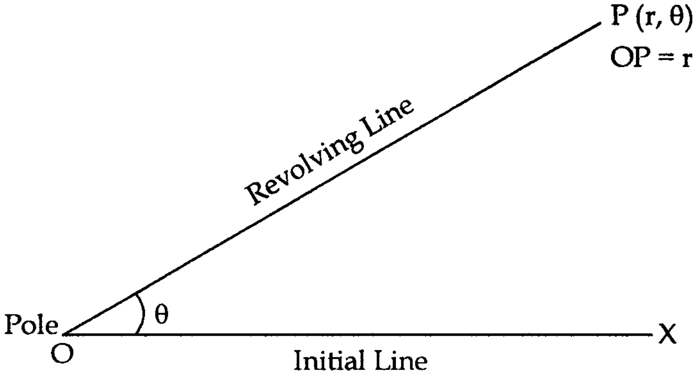
(iv) There is no asymptote which is parallel to either axis but there is an oblique asymptote $x + y + a = 0$.
(v) x and y both cannot be negative, because that will make L.H.S of the curve negative and R.H.S positive. This implies no portion of the curve lies in the third quadrant.
Shape of the curve given below.



**Example 5 :** Trace the curve
$a^2 x^2 = y^3 (2a - y)$
**Solution :** (i) symmetry about y-axis as the power of x are even.
(ii) $x = 0$ we get $y = 0$ and $2a$, ∴ (0, 0) and (0, 2a) are the points; when $y = 0$ we get $x = 0$, ∴ (0, 0)
(iii) T(0, 0) are given by $a^2 x^2 = 0$ i.e. $x = 0$ i.e. y-axis.
(iv) No asymptotes.
(v) when y is either greater than 2a or negative, then $x^2$ is negative and hence x will be imaginary. Hence no part of curve lies below $y = 0$ i.e. x-axis and beyond $y = 2a$ with the above data the shape of the curve is as shown in figure given below.



**Polar Co-ordinates :** If we have any horizontal line OX called the initial line and another line called the revolving line makes an angle θ with the initial line then the polar co-ordinates of a point P on it where OP = r are (r, θ).



The point O is called the pole and the angle θ is called the vectorial angle of the point P and the length r is called the radius vector.

**Rules for tracing Polar Curves -**

**I. Symmetry**

If in the given equation θ be changed into -θ and the equation of the curve remains unchanged, then there is symmetry about the initial line.

If the equation of the curve remains unchanged when r is changed into -r, then there is symmetry about the pole.

**II. Plotting the points** Give to θ certain value and find the corresponding values of r and then plot points. Sometimes it is inconvenient to find the corresponding values of r for certain values of θ and even if we find they involve radical, then in such cases we should consider a particular region for θ and as certain whether r increases or decreases in that region.

For this we must remember the following :

| At $\theta = 0$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|
| $\sin\theta$= Min | Max | Min | Max (in Magnitude) | Min |
| $\cos\theta$ = Max | Min | Max (in Magnitude) | Min | Max |

e.g. $r = a(1 + \cos\theta)$

| $\theta = 0$ | $60^0$ | $90^0$ | $120^0$ | $180^0$ |
|---|---|---|---|---|
| $r = 2a$ | $\frac{3a}{2}$ | $a$ | $\frac{a}{2}$ | $0$ |

**III. Region** No part of the curve shall exist for these values of θ which make corresponding values of r imaginary.

e.g. $r^2 = a^2 \cos^2\theta$

As θ increases from $45^0$ to $135^0$, then 2θ increases from $90^0$ to $270^0$ and this interval cos2θ will always be -negative and consequently $r^2$ will also be negative, which shall give imaginary values of r. Hence no part of the curve shall exist between θ = $45^0$ and θ = $135^0$.

We also find the limits to the values of r.

i.e. $r = a \sin 2\theta$

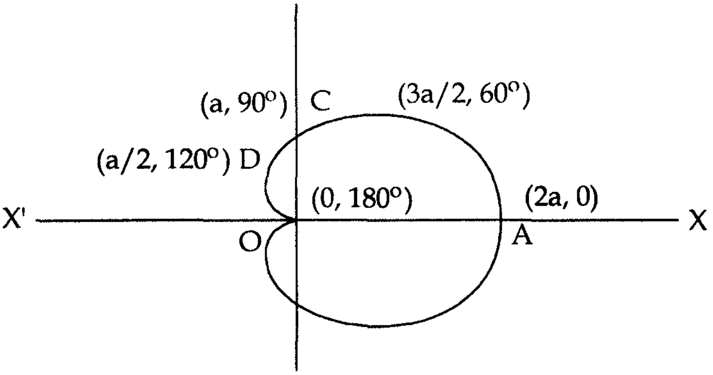
Now whatever θ may be sin2θ can never be greater than unity and hence a sin2θ or shall never be greater then a i.e. the curve shall entirely lie within the circle r =a.

**Example 6 :** Trace the curve

$r = a(1+\cos\theta)$ (Cardiod)

**Solution :** If we change θ into -θ, the equation of the curve remains unchanged and hence there is symmetry about the initial line.

| θ =0 | $60^0$ | $90^0$ | $120^0$ | $180^0$ |
|---|---|---|---|---|
| r = 2a | $\frac{3a}{2}$ | a | $\frac{a}{2}$ | 0 |

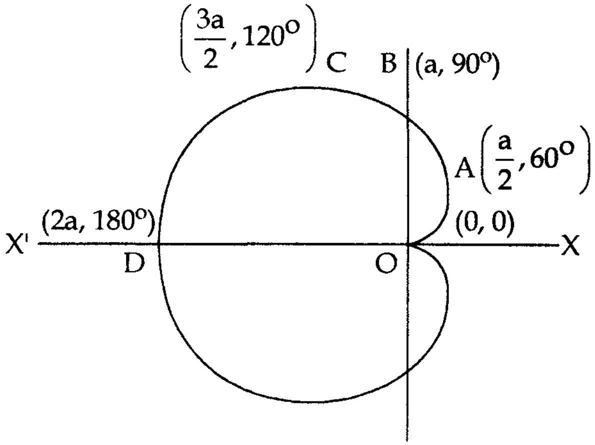


We observe that r continually goes on decreasing as θ increases from $0^0$ to $180^0$. The above points trace the curve above the initial line and the other portion is drawn by symmetry. With the above data the shape of the curve is as in the figure above.

**Example 7 :** Trace

$r = a(1-\cos\theta)$ (Cardiod)

**Solution :** Trace as above



$\left(\frac{3a}{2}, 120^o\right)$
C
B
(a, 90°)
$A\left(\frac{a}{2}, 60^o\right)$
(2a, 180°)
(0, 0)
X'
D
O
X

**Example 8 :** Trace the curve $r^2 = a^2 \cos 2\theta$
(Lemniscate of Bernoulli)

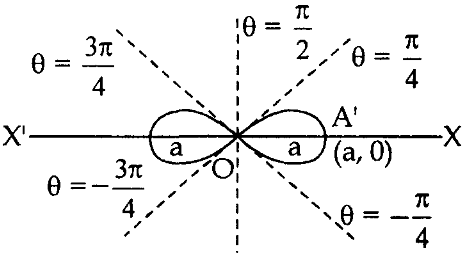
(U.P.T.U. 2008-09. 2001)

**Solution :** Symmetry about the initial line putting r = 0, $\cos 2\theta = 0$ or $2\theta = \pm\frac{\pi}{2}$ or $\theta = \pm\frac{\pi}{4}$. Hence the straight lines $\theta = \pm\frac{\pi}{4}$ are the tangents to the curve at the pole.

As θ varies from 0 to π, r varies as given below : -

| $\theta = 0^0$ | $30^0$ | $45^0$ | $90^0$ | $135^0$ | $150^0$ | $180^0$ |
|---|---|---|---|---|---|---|
| $r^2 = a^2$ | $\frac{a^2}{2}$ | 0 | $-a^2$ | 0 | $\frac{a^2}{2}$ | $a^2$ |
| $r = \pm a$ | $\pm\frac{a}{\sqrt{2}}$ | 0 | imaginary | 0 | $\pm\frac{a}{\sqrt{2}}$ | $\pm a$ |

The above data shows that curve does not exist for values of θ lying between $45^0$ and $135^0$. With the above data the shape of the curve is shown as given below.



$\theta = \frac{\pi}{2}$
$\theta = \frac{3\pi}{4}$
$\theta = \frac{\pi}{4}$
A'
X'
a
a
(a, 0)
X
O
$\theta = -\frac{3\pi}{4}$
$\theta = -\frac{\pi}{4}$

**Example 9 :** Trace the curve

r = a cos2θ

(U.P.T.U. 2003)

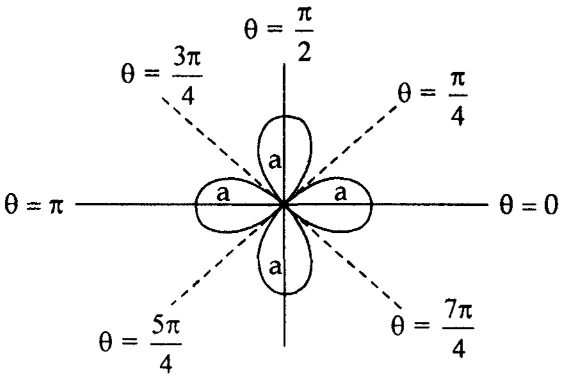
**Solution :** Symmetry about the initial line putting r =0, cos2θ=0 or $2\theta = \pm\frac{\pi}{2}$ or

$\theta = \pm\frac{\pi}{4}$. i.e. the straight lines $\theta = \pm\frac{\pi}{4}$ are the tangents to the curve at the pole.

Corresponding values of θ and r are given below.

| θ = 0° | 30° | 45° | 60° | 90° | 120° | 135° | 150° | 180° |
|---|---|---|---|---|---|---|---|---|
| r = a | $\frac{a}{2}$ | 0 | $-\frac{a}{2}$ | -a | $-\frac{a}{2}$ | 0 | $\frac{a}{2}$ | a |

Plot these points and due to symmetry about initial line the other portion can be traced.



**Note :** The curve is of the form of r = a cos nθ (or r = a sin nθ) and in such case there will be n or 2n equal loops according as n is odd or even.

**Example 10 :** Trace the curve

r = a cos3θ (U.P.P.C.S. 2004)

Symmetry about the initial line. Putting r =0, we get cos3θ =0

or $3\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$

or $\theta = \pm\frac{\pi}{6}, \pm\frac{\pi}{2}, \pm\frac{5\pi}{6}$

Which give the tangents at the pole

$\because \frac{dr}{d\theta} = -a\sin 3\theta$

$\therefore$ equating $\frac{dr}{d\theta}$ to zero we get $\sin 3\theta = 0$

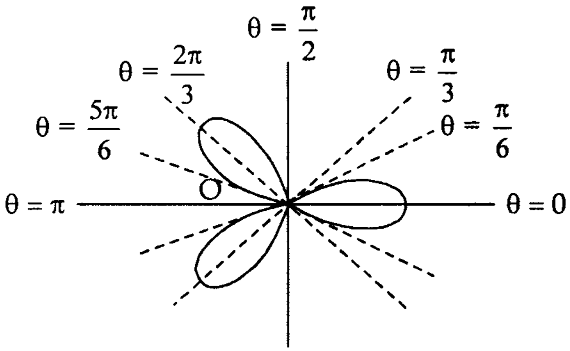
or $3\theta = 0, \quad \pi, \quad 2\pi, \quad 3\pi, \quad 4\pi$

or $\theta = 0, \quad \frac{\pi}{3}, \quad \frac{2\pi}{3}, \quad \pi, \quad \frac{4\pi}{3}$

Which give the maximum values of θ and are given below : -

| $\theta = 0$ | $\frac{\pi}{6}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{5\pi}{6}$ | $\pi$ |
|---|---|---|---|---|---|---|
| $r = a$ | 0 | $-a$ | 0 | $a$ | 0 | $-a$ |

plot the above points and with the above data the shape of the curve is a shown in figure given below :



**Example 11 :** Trace the curve $r = a \sin 2\theta$ (U.P.T.U. 2001)

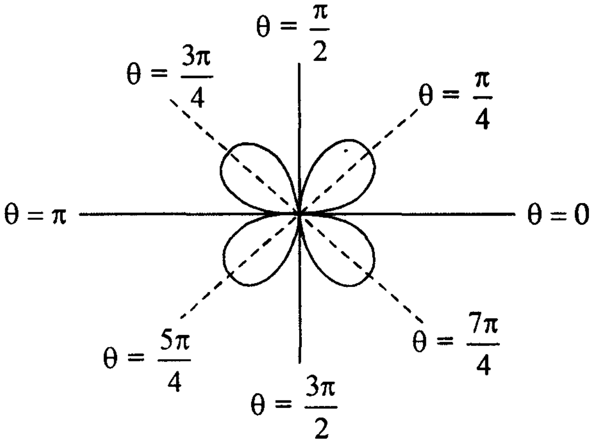
**Solution :** Symmetry about the line $\theta = \frac{\pi}{2}$ putting $r = 0$, $\sin 2\theta = 0$

or $2\theta = 0, \quad \pi, \quad 2\pi, \quad 3\pi$

or $\theta = 0, \quad \frac{\pi}{2}, \quad \pi, \quad \frac{3\pi}{2}$

Which are the tangent at the pole.

$\frac{dr}{d\theta} = 2a\cos 2\theta$, Equating $\frac{dr}{d\theta}$ to zero, we get $\cos 2\theta = 0$


$\theta = \frac{\pi}{2}$
$\theta = \frac{3\pi}{4}$
$\theta = \frac{\pi}{4}$
$\theta = \pi$
$\theta = 0$
$\theta = \frac{5\pi}{4}$
$\theta = \frac{7\pi}{4}$
$\theta = \frac{3\pi}{2}$

or $2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

or $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Which gives the value of r and which is equal to a sin $\left(2\frac{\pi}{4}\right) = a\sin\frac{\pi}{2} = a$.

The corresponding values of θ and r are given below.

| $\theta = 0$ | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $\frac{3\pi}{4}$ | $\pi$ | $\frac{5\pi}{4}$ | $\frac{3\pi}{2}$ | $\frac{7\pi}{4}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|---|
| $r = 0$ | a | 0 | a | 0 | a | 0 | a | 0 |

Plot the above points and with above data the shape of the curve is shown as above.

**Parametric Curves**

**Example 9 :** Trace the curve

$x = a(t + \sin t), y = a(1 - \cos t)$ Cycloid

**Solution :** The given curves are $x = a(t + \sin t), y = a(1 - \cos t)$

so $\frac{dx}{dt} = a(1 + \cos t)$ and $\frac{dy}{dt} = a \sin t$

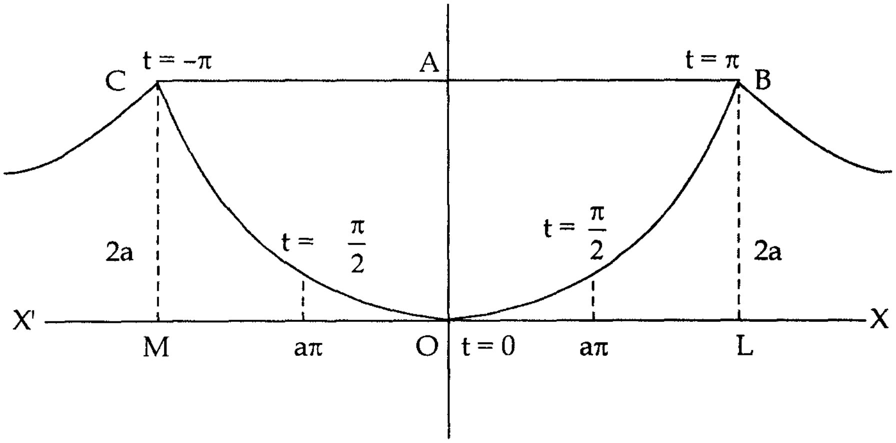
$$\therefore \frac{dy}{dt} = \frac{a\sin t}{a(1+\cos t)}$$

$$= \frac{2\sin t/2 \cos t/2}{2\cos^2 t/2}$$

$= \tan t/2$

| t =0 | $\frac{\pi}{2}$ | $\pi$ | $-\pi/2$ | $-\pi$ |
|---|---|---|---|---|
| x =0 | $a\left(\frac{\pi}{2}+1\right)$ | $a\pi$ | $-a\left(\frac{\pi}{2}+1\right)$ | $-a\pi$ |
| y =0 | a | 2a | a | 2a |
| $\frac{dy}{dx}=0$ | 1 | $\infty$ | -1 | $-\infty$ |

we observe that there is a point (0, 0) on the curve and slope of the tangent there is 0 i.e. tangent will be x axis. Again we find that there is a point $\left[a\left(\frac{\pi}{2}+1\right), a\right]$ and slope being equal to 1 showing that tangent at that point will make an angle of $45^0$ with positive direction of x axis. Further there is a point (aπ, 2a) and slope being ∞ indicates that tangent makes an angle of $90^0$ with x axis. Similarly other points are observed. If we go on giving to t values greater than π or less than -π, we shall have the same type of branches.



The point O is called vertex and X'OX i.e. LM is tangent at the vertex, CAB is called the base.

**Example 10 :** Trace the curve
x = a (t - sin t), y = a(1 - cos t)

**Solution :**

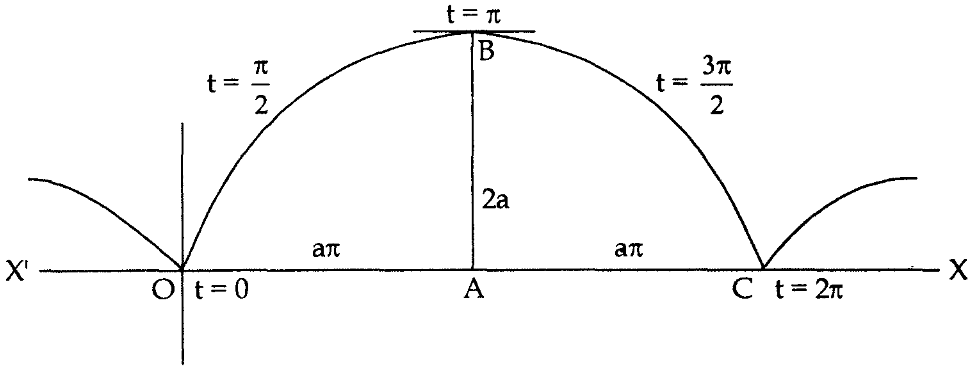
$$\frac{dx}{dt} = a(1-\cos t), \frac{dy}{dt} = a\sin t$$

$$\therefore \frac{dy}{dx} = \frac{a\sin t}{a(1-\cos t)} = \frac{2\sin t/2 \cos t/2}{2\sin^2 t/2}$$

$= \cot t/2$

| $t = 0$ | $\pi/2$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|
| $x = 0$ | $a\left(\frac{\pi}{2} - 1\right)$ | $a\pi$ | $a\left(\frac{3\pi}{2} + 1\right)$ | $2a\pi$ |
| $y = 0$ | $a$ | $2a$ | $a$ | $0$ |
| $\frac{dy}{dx} = \infty$ | $1$ | $0$ | $-1$ | $\infty$ |

Here we have taken t from 0 to 2π and we get one complete cycloid. If we give negative values we shall get the corresponding cycloid on the other side of y axis and so on.



**Example 11 :** Trace the curve

$x = a \cos t + \frac{a}{2} \log \tan^2 \frac{t}{2}$ , $y = a \sin t$ (Tractrix)

**Solution :**

$$\frac{dy}{dt} = -a\sin t + \frac{a}{2}2.\frac{1}{\tan\frac{t}{2}}.\frac{1}{2}\sec^2\frac{t}{2}$$

$$= -a\sin t + \frac{a}{2\sin\frac{t}{2}\cos\frac{t}{2}}$$

$$= \frac{a}{\sin t}(1-\sin^2 t)$$

$$= a\frac{\cos^2 t}{\sin t}$$

and $\frac{dx}{dt} = a\cos t$

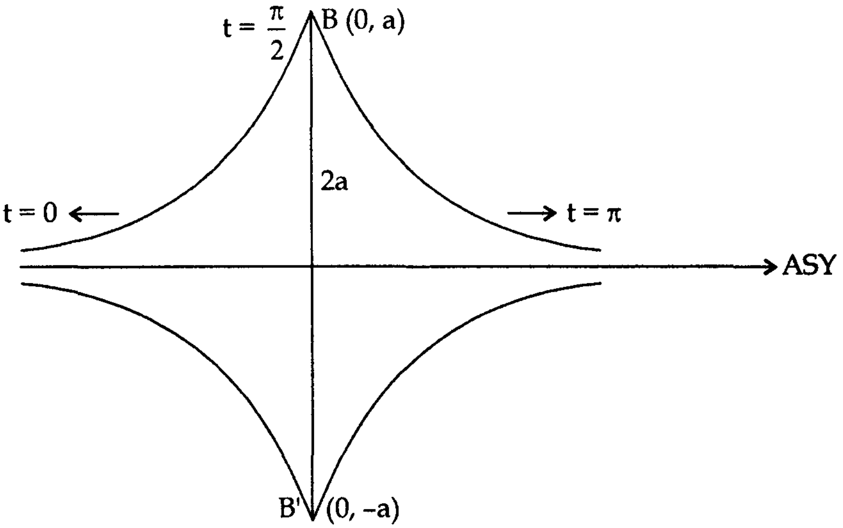
$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= a\cos t.\frac{\sin t}{a\cos^2 t}$$

= tan t

| t =0 | $\frac{\pi}{2}$ | $\pi$ | $-\frac{\pi}{2}$ |
|---|---|---|---|
| $x = -\infty$ | 0 | $\infty$ | 0 |
| $y = 0$ | a | 0 | -a |
| $\frac{dy}{dx} = 0$ | $\infty$ | 0 | $-\infty$ |

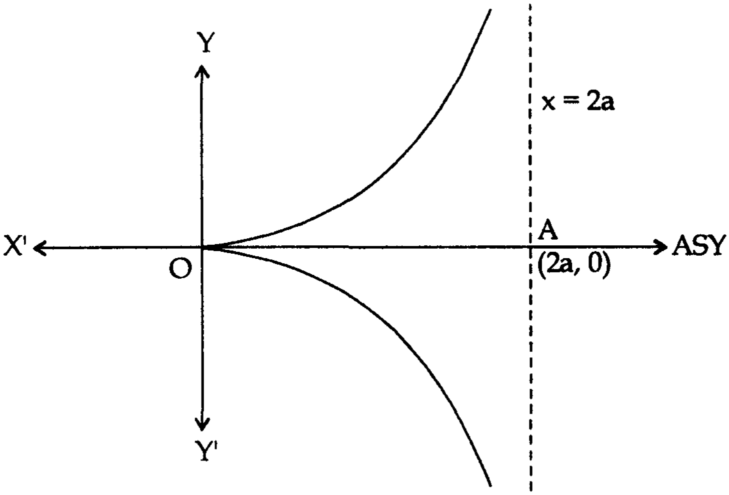
When t =0 the corresponding point is (- ∞, 0) i.e. a point at infinity on the negative side of x axis and slope there, being 0 indicates x axis will be tangent at that point which is situated at ∞ hence x axis is an asymptotes. Similarly plot other points.



**EXERCISE**

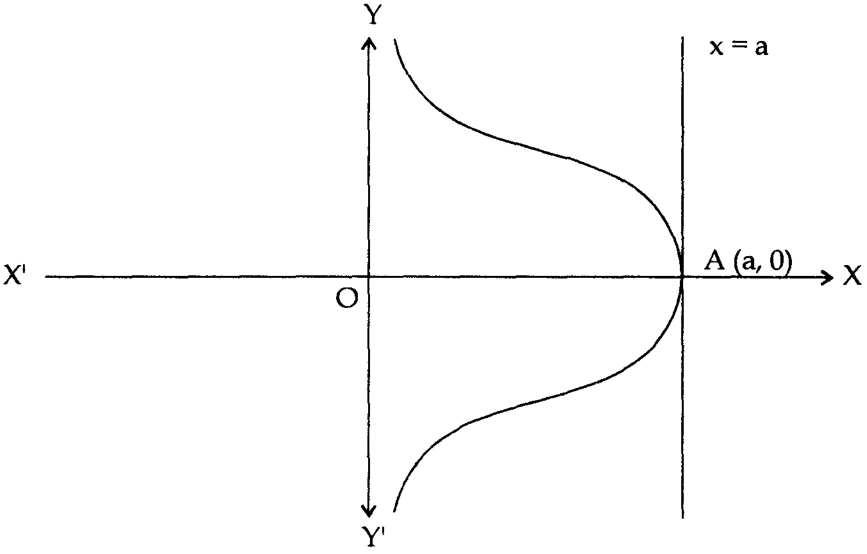
1. Trace the curve $y^2 (2a - x) = x^3$ (Cissoid)

Ans.



2. Trace the curve $xy^2 = a^2(a - x)$

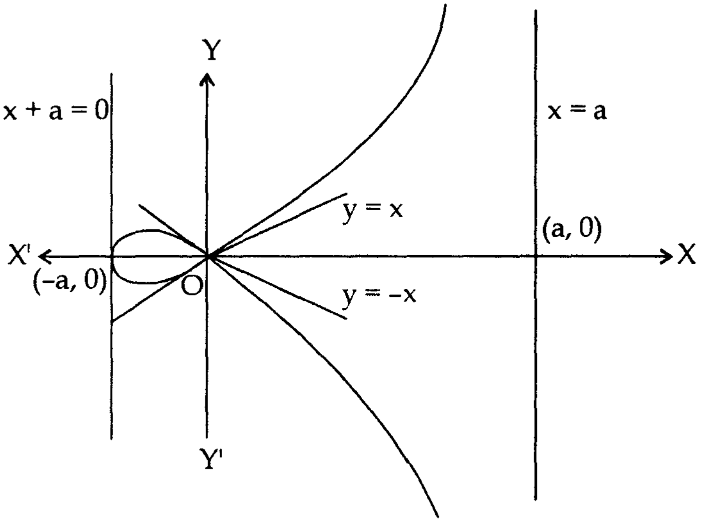
Ans.



3. Trace the curve $y^2(a - x) = x^2(a + x)$ (Strophoid)

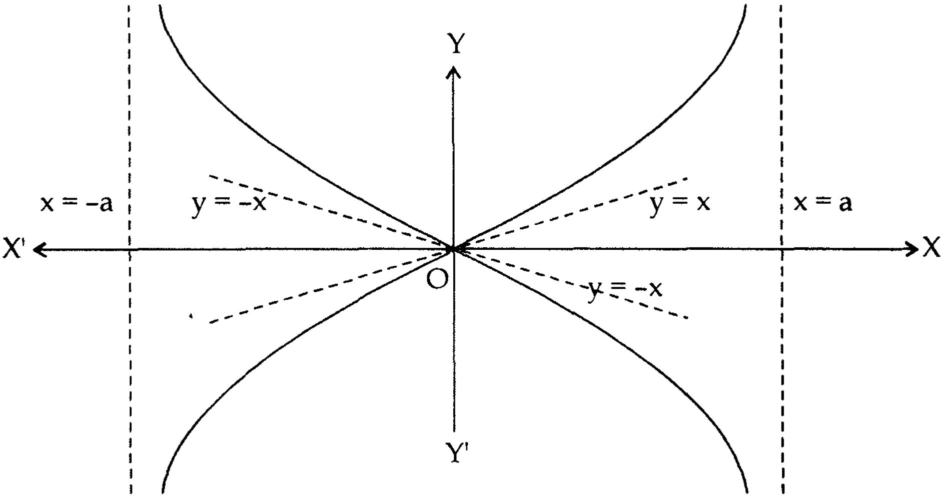
(U.P.P.C.S. 1994)

Ans.

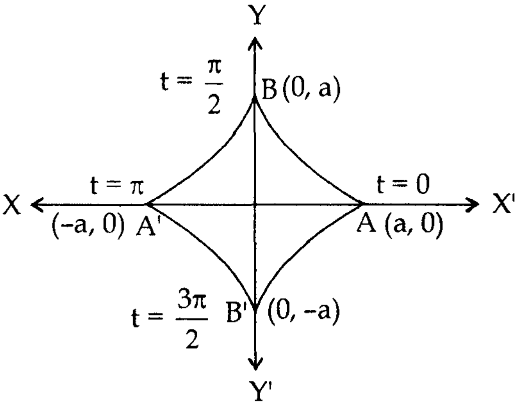


4. Trace the curve $x^2y^2 = a^2(y^2 - x^2)$

Ans.



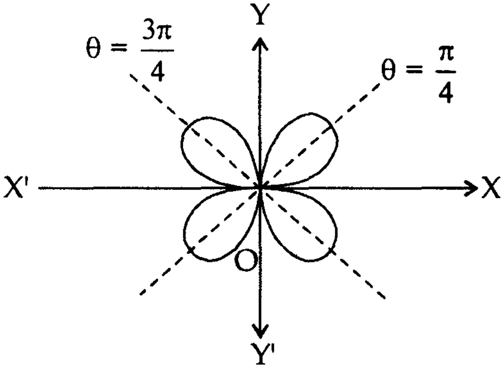
5. Trace the curve $x^{2/3}+y^{2/3} = a^{2/3}$ (Astroid)
Ans.



6. Trace the curve r = a sin2θ

(U.P.T.U. 2002)

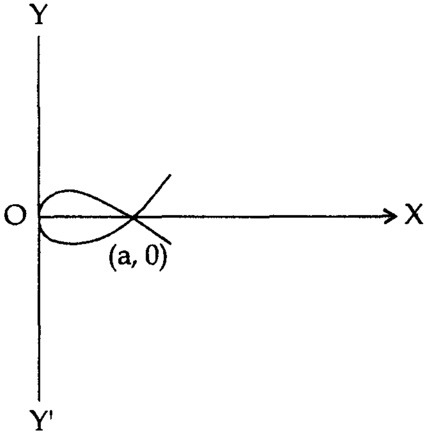
Ans.



Remark : In the curve r = a sin nθ or r = a cos nθ, ∃ n loops if n is odd and 2n if n is even.

7. Trace the curve $3ay^2 = x(x - a)^2$

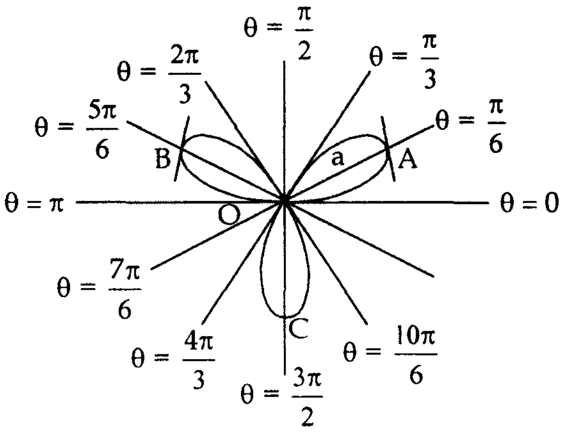
Ans.



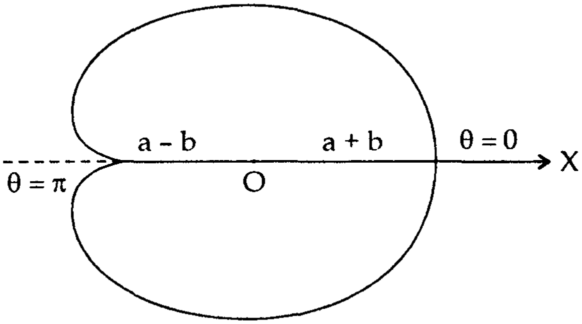
8. Trace the curve r = a sin 3θ

(U.P.T.U. 2001)

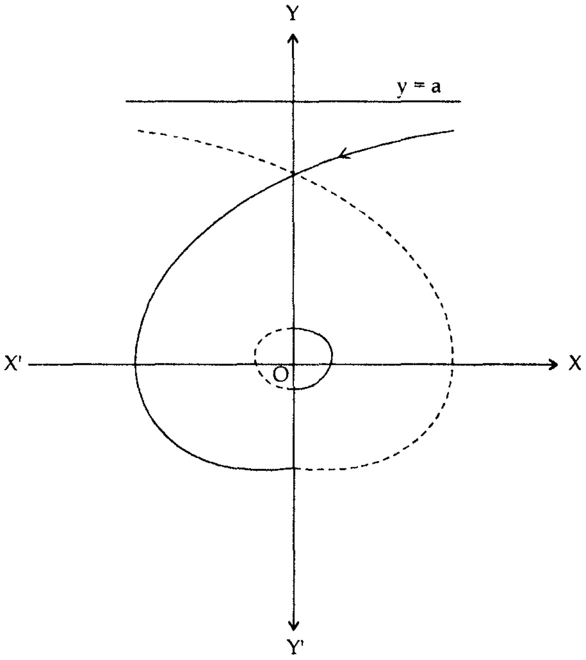
Ans.


$\theta = \frac{\pi}{2}$
$\theta = \frac{2\pi}{3}$
$\theta = \frac{\pi}{3}$
$\theta = \frac{5\pi}{6}$
$\theta = \frac{\pi}{6}$
B
a
A
$\theta = \pi$
$\theta = 0$
O
$\theta = \frac{7\pi}{6}$
C
$\theta = \frac{4\pi}{3}$
$\theta = \frac{10\pi}{6}$
$\theta = \frac{3\pi}{2}$

9. Trace the curve $r = a+b\cos\theta$, $a > b$
(Pascal's Limacon)
Ans.

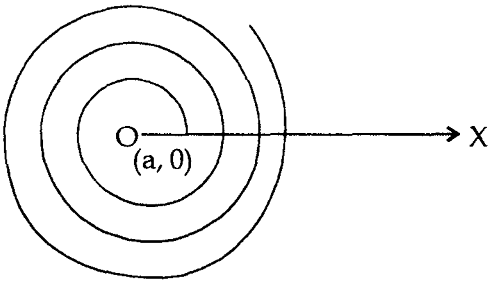


10. Trace the curve reciprocal spiral $r\theta = a$
Ans.



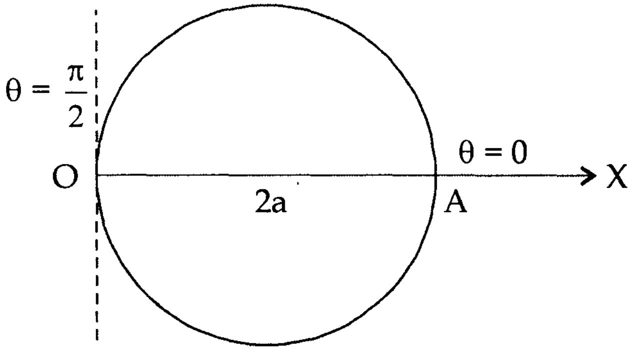
11. Trace the equiangular spiral $r = a\ e^{\theta\cot\alpha}$

Ans.



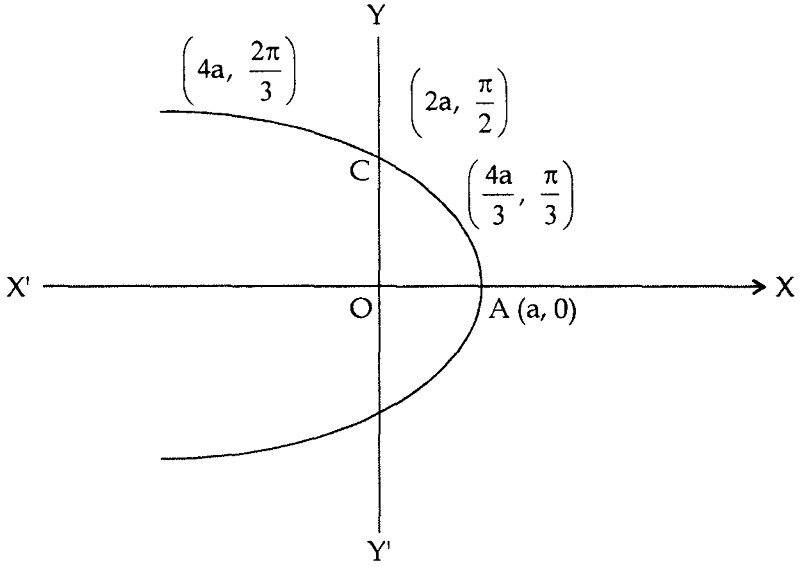
12. Trace the curve $r = 2\ a \cos\theta$ (circle)

Ans.



13. Trace the curve $\frac{2a}{r} = 1 + \cos\theta$ (Parabola)

Ans.



Y
$\left(4a, \frac{2\pi}{3}\right)$
$\left(2a, \frac{\pi}{2}\right)$
C
$\left(\frac{4a}{3}, \frac{\pi}{3}\right)$
X'
O
A (a, 0)
X
Y'

**Objective problems**

Four alternative answers are given for each question, only one of them is correct. Tick mark the correct answer -

1. Which of the following lines is a line of symmetry of the curve $x^3 + y^3 = 3(xy^2 + yx^2)$?

(I.A.S. 1993)

(i) $x = 0$ (ii) $y = 0$

(iii) $y = x$ (iv) $y = -x$

Ans. (iii)

2. For the curve $y^2 (2a - x) = x^3$, which of the following is false ?
(i) The curve is symmetrical about x axis.
(ii) The origin is a cusp.
(iii) $x = 2a$ is a asymptote.
(iv) As $x \to 3a, y \to \infty$ .

Ans. (iv)

3. The curve $x^2(x^2 + y^2) = a^2 (x^2 - y^2)$
(1) Symmetrical about $x = 0, y = 0$
(2) $y = \pm x$ are tangents at origin
(3) $x = \pm a$ are the asymptotes.
(4) has no point of inflexion of these statements -
(i) Only 1 and 2 are correct.
(ii) 1, 2 and 4 are correct.
(iii) 2, 3 and 4 are correct.
(iv) All are correct.

Ans. (ii)

4. The equations of the inverted cycloid is -

(U.P.P.C.S. 1994)

(i) $x = a(\theta - \cos\theta)$
$y = a(1 - \cos\theta)$
(ii) $x = a(\theta + \sin\theta)$
$y = a(1 - \cos\theta)$
(iii) $x = a(\theta - \sin\theta)$
$y = a(1 - \cos\theta)$
(iv) $x = a(\theta - \sin\theta)$
$y = a(1 + \cos\theta)$

Ans. (iii)

5. The curve given by the equations $x = a \cos^3\theta$, $y = a \sin^3\theta$ is symmetric about -

(U.P.P.C.S. 1995)

(i) Both the axes.
(ii) x - axis only.
(iii) y - axis only.
(iv) None of the two axes.

Ans. (i)

6. The curve $x^3 + y^3 - 3axy = 0$ has at origin -
(i) Node (ii) Cusp of first species
(iii) Cusp of second species (iv) Conjugate point

Ans. (i)

(R.A.S. 1994)

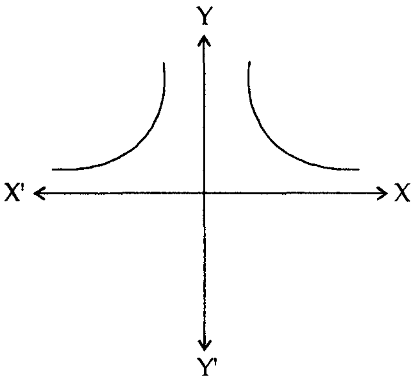
7. The curve $y = 3x^5 - 30x^4 + 3x - 20$ has how many points of inflexion?

(i) None (ii) One
(iii) Two (iv) Four

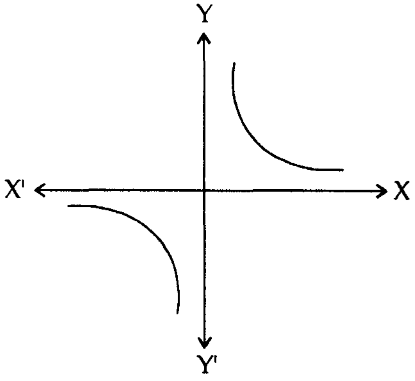
Ans. (ii)
(M.P.P.C.S. 1995)

8. The graph of the function $f(x) = \frac{1}{x}$ is

(i)



(ii)



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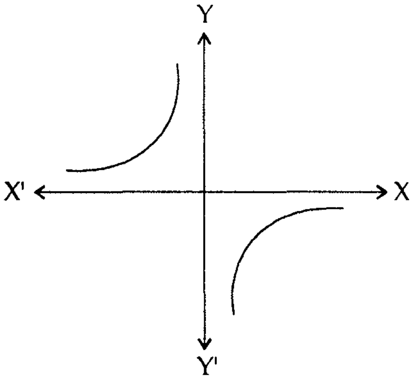
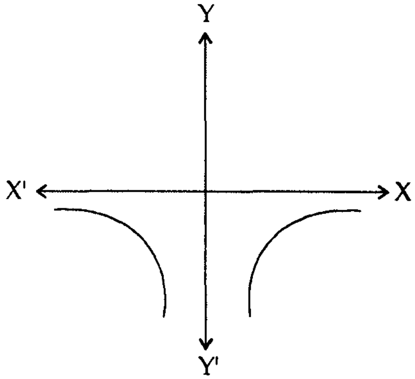


(iv)



Ans. (ii)
(R.A.S. 1995)

9. Consider the following statements with regard to the curve $x^6 + y^6 = a^2x^2y^2$
Assertion (A) - Curve is symmetrical about both the axes.
Reason (R) - By putting $-x$ and $-y$ in place of x and y respectively, the equation of the curve remains unaltered.
of these statements -
(i) Both A and R are true and R is a correct explanation of A.
(ii) Both A and R are true and R is not a correct explanation of A.
(iii) A is true but R is false.
(iv) A is false but R is true.

Ans. (ii)

10. The graph of the curve $x = a\left(\cos t + \frac{1}{2}\log\tan^2\frac{t}{2}\right)$, y = a sin t is symmetrical about the line.

(i) x =0 (ii) y =0
(iii) y = x (iv) None of these.

Ans. (ii)

11. The graph of $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$ is a

(i) Circle (ii) Ellipse
(iii) Cycloid (iv) None of these.

Ans. (i)

12. If x and y both are the odd functions of t then the curve is symmetrical -

(i) About x axis (ii) About y axis
(iii) About y = x (iv) In opposite quadrant

Ans. (iv)

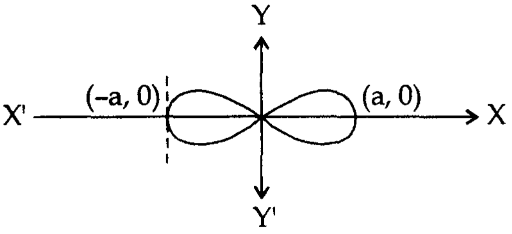
13. The curve x = a cos$\phi$, y = a sin$\phi$ is

(i) Circle (ii) Ellipse
(iii) Parabola (iv) None of these

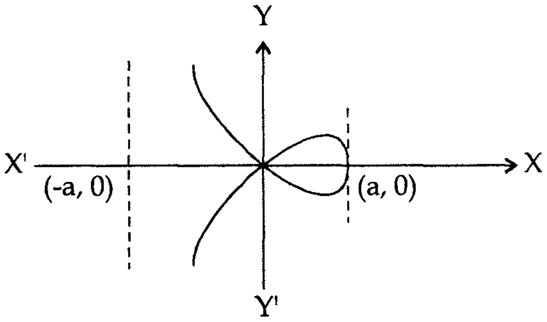
Ans. (i)

14. The curve traced by the equation $y^2(a + x) = x^2(a - x)$ is

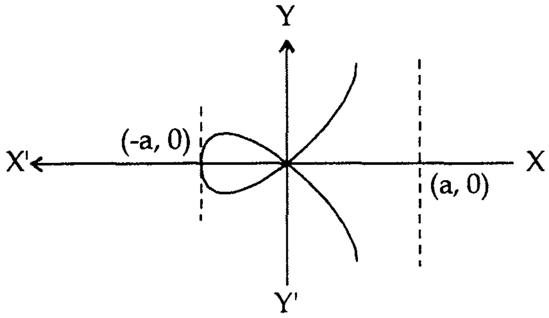
(i)



(ii)



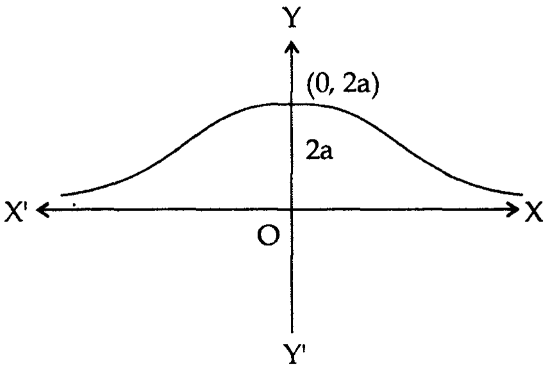
(iii)



(iv) None of these.

Ans. (ii)

15. The curve traced below is represented by -

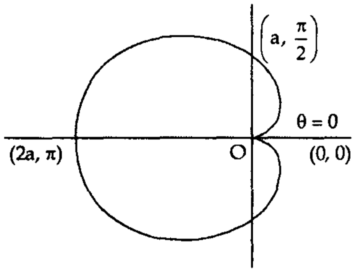


(i) $x(y^2 + 4a^2) = 8a^3$ (ii) $xy^2 = 4a^2(2a - x)$
(iii) $y(x^2 + 4a^2) = 8a^3$ (iv) None of these

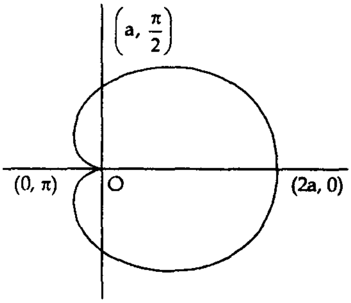
Ans. (iii)

16. The curve traced by $r = a(1 + \cos\theta)$ is

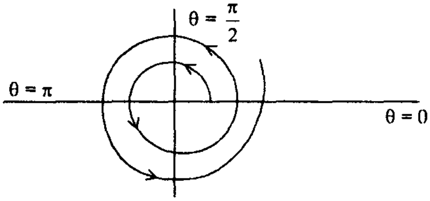
(i)



(ii)



(iii)

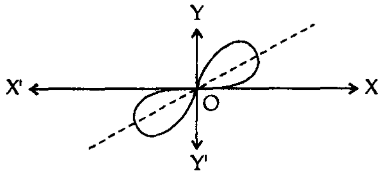


(iv) None of these.

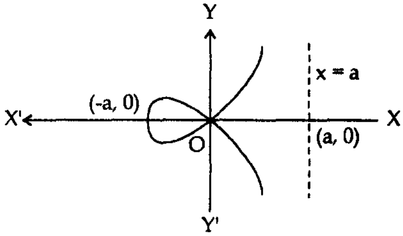
Ans. (ii)

17. The curve traced by $x = \dfrac{3at}{1+t^3}$, $y = \dfrac{3at^2}{1+t^2}$ is

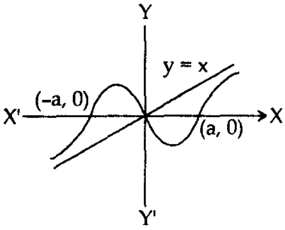
(i)



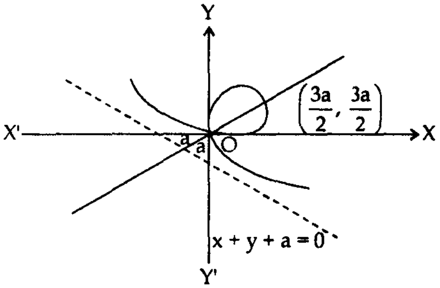
(ii)



(iii)



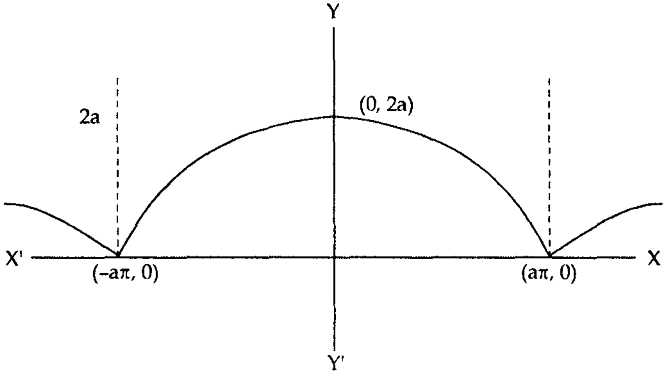
(iv)



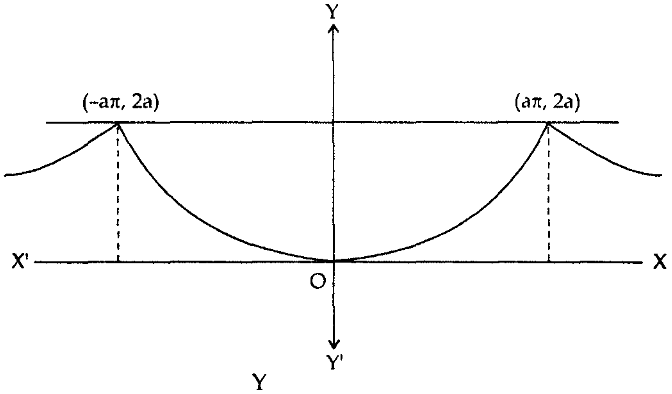
Ans. (iv)

18. The curve traced by $x = a(t + \sin t)$ and $y = a(1 + \cos t)$ is -

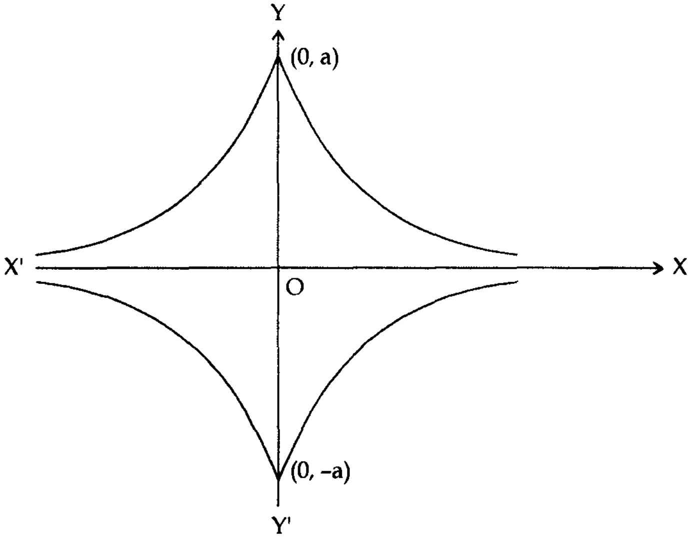
(i)



(ii)



(iv)



(iv) None of these

Ans. (i)

19. The Folium of Descartes is given by the equation
(i) $(x^2 + y^2)^2 = a^2(x^2 - y^2)$
(ii) $x^3 + y^3 = a^3$
(iii) $x^3 + y^3 = 3axy$
(iv) None of these

Ans. (iii)

20. The Lemniscate of Bernoulli is represented by the equation
(i) $x^6 + y^6 = 3x^2y^2$
(ii) $(x^2 + y^2)^2 = a^2(x^2 - y^2)$
(iii) $x^5 + y^5 = 5a^2x^2y^2$
(iv) None of these

Ans. (ii)

21. The number of leaves in the curve $r = a \sin 5\theta$ are

| | |
|---|---|
| (i) Two | (ii) Five |
| (iii) Ten | (iv) None of these. |

Ans. (ii)

22. Match the list I with list II -
List I
curves
(a) $x^2y^2 = x^2 - a^2$
(b) $y(x^2 + 4a^2) = 8a^3$
(c) $y^2x = a^2(x - a)$
(d) $x^2y^2 = a^2(x^2 + y^2)$
List II
Portion of the curve
(1) Lies between $y = 0$ and $y = 2a$.
(2) Lies outside the lines $x = \pm a$ and inside the lies $y = \pm 1$.
(3) Does not lie between the lines $x = \pm a$, $y = \pm a$.
(4) Does not lie between the lines $x = 0$ and $x = a$.
Correct match is -

| | a | b | c | d |
|---|---|---|---|---|
| (i) | 2 | 4 | 1 | 3 |
| (ii) | 2 | 1 | 4 | 3 |
| (iii) | 2 | 1 | 3 | 4 |
| (iv) | 1 | 2 | 4 | 3 |

Ans. (ii)

23. Match the list I with list II.
List I
curves
(a) $y^2(a^2 + x^2) = x^2(a^2 - x^2)$

(b) $a^4y^2 = a^2 x^4 - x^6$

(c) $x^2 y^2 = (a + y)^2 (a^2 - y^2)$

(d) $x^3 - ay^2 = 0$

(e) $x^3 + y^3 - 3axy = 0$

List II.

Symmetry about

(1) $y = x$

(2) $x = 0$

(3) $x = 0, y = 0$

(4) $y = 0$

Then correct match is

| | a | b | c | d | e |
|---|---|---|---|---|---|
| (i) | 3 | 1 | 2 | 4 | 3 |
| (ii) | 3 | 3 | 2 | 1 | 4 |
| (iii) | 3 | 3 | 2 | 4 | 1 |
| (iv) | 3 | 2 | 3 | 4 | 1 |

Ans. (iii)

24. Match the list I with list II.

List I.

(a) $r = a(1 + \cos\theta)$

(b) $r = a(1 + \sin\theta)$

(c) $r^2 = a^2 \cos 2\theta$

(d) $r = a \cos 2\theta$

List II

Symmetrical about

(1) $\theta = 0$ and $\theta = \pi/2$

(2) $\theta = 0$

(3) $\theta = \frac{\pi}{2}$

The correct match is

| | a | b | c | d |
|---|---|---|---|---|
| (i) | 2 | 3 | 1 | 1 |
| (ii) | 2 | 1 | 3 | 1 |
| (iii) | 2 | 3 | 3 | 1 |
| (iv) | 3 | 2 | 1 | 2 |

Ans. (i)

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# Chapter 4

# Expansion of Function of One Variable

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**Taylor's Theorem :** If $f(x+h)$ is a function of h which can be expanded in ascending powers of h and is differentiable term by term any number of times w.r.t.h then

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\lfloor 2} f''(x) + \frac{h^3}{\lfloor 3} f'''(x) + \ldots\ldots\ldots\ldots\ldots \frac{h^n}{\lfloor n} f^n(x) + \ldots$$

**Proof :** Let $f(x+h) = A_0 + A_1h + A_2h^2 + A_3h^3 + A_4h^4 + \ldots\ldots\ldots\ldots\ldots\ldots(i)$

Differentiating it successively w.r.t.h, we get

$f'(x+h) = A_1 + 2A_2h + 3A_3h^2 + 4A_4h^3 + \ldots\ldots\ldots\ldots\ldots\ldots(ii)$

$f''(x+h) = 2A_2 + 3.2A_3h + 4.3A_4h^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots(iii)$

$f'''(x+h) = 3.2\,A_3 + 4.3.2\,A_4h + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(iv)$

Putting h =0 in (i), (ii), (iii), (iv), ............etc. we get

$$A_0 = f(x),\ A_1 = \frac{1}{\lfloor 1} f'(x),\ A_2 = \frac{1}{\lfloor 2} f''(x).\ A_3 = \frac{1}{\lfloor 3} f'''(x) \text{ etc.}$$

Substituting these values of $A_0, A_1, A_2$, ..........etc. in (i) we get

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\lfloor 2} f''(x) + \frac{h^3}{\lfloor 3} f'''(x) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots(v)$$

**Corollary 1.** Substituting x =a in (v), we get

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2} f''(a) + \frac{h^3}{\lfloor 3} f'''(a) + \ldots\ldots\ldots\ldots\ldots(vi)$$

**Corollary 2.** Substituting a =0 and h= x in (vi), we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \ldots\ldots\ldots$$

Which is Maclaurin's theorem,

**Corollary 3.** Substituting h = x -a in (vi), we get

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{\lfloor 2} f''(a) + \frac{(x-a)^3}{\lfloor 3} f'''(a) + \ldots\ldots\ldots$$

This is the expansion of f(x) in powers of (x - a).

**Example 1 :** Show that $\log(x+h) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} - \ldots\ldots\ldots\ldots$

**Solution :** Here we are to expand in powers of x. Thus we have to use the form

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{\lfloor 2} f''(h) + \frac{x^3}{\lfloor 3} f'''(h) + \ldots\ldots\ldots \text{(i)}$$

Here f(x+h) = log (x +h)

$$\therefore f(h) = \log h;\ f'(h) = \frac{1}{h};\ f''(h) = -\frac{1}{h^2}\ ;\ f'''(h) = \frac{2}{h^3} \text{ etc.}$$

∴ Substituting these values in (i) we get

$$\log(x+h) = \log h + x\left(\frac{1}{h}\right) + \frac{x^2}{\lfloor 2}\left(-\frac{1}{h^2}\right) + \frac{x^3}{\lfloor 3}\left(\frac{2}{h^3}\right) + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} + \ldots\ldots\ldots\ldots\ldots\ldots$$

Hence Proved

**Example 2:** Express the polynomial $2x^3 - 7x^2 + x - 1$ in powers of (x-2)

(B.P.S.C. 2007)

**Solution :**

Here $f(x) = 2x^3 + 7x^2 + x - 1$

$\therefore f'(x) = 6x^2 + 14x + 1;\ f''(x) = 12x + 14\ f'''(x) = 12;\ f^{iv}(x) = 0$

Here we are to use the following Taylor's theorem

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{\lfloor 2} f''(a) + \frac{(x-a)^3}{\lfloor 3} f'''(a) + \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Here a = 2, as we are to expand in powers of x-2

∴ putting x =2 in f(x), f'(x), ...........etc, we get

$f(2) = 2(2)^3 + 7(2)^2 + 2 - 1 = 45$

$f'(2) = 6(2)^2 + 14(2) + 1 = 53$

$f''(2) = (12)(2) + 14 = 38$

$f'''(2) = 12,\ f^{iv}(2) = 0$ etc.

∴ From (i) substituting a =2 and these values, we get

$$2x^3 + 7x^2 + x - 1 = 45 + (x-2)\,53 + \frac{(x-2)^2}{\lfloor 2}(38) + \frac{(x-2)^2}{\lfloor 3}(12)$$

$= 45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$ Answer.

**Example 3:** Expand sin x in powers of $\left(x - \frac{\pi}{2}\right)$.

**Solution:** Here f(x) = sin x

$$= \sin\left\{\frac{\pi}{2} + \left(x - \frac{\pi}{2}\right)\right\}$$

Here we should use the following form of Taylor's theorem.

$$f(x) = f(a) + (x - a)\,f'(a) + \frac{(x-a)^2}{\lfloor 2}\,f''(a) + \frac{(x-a)^3}{\lfloor 3}\,f'''(a) + \ldots\ldots\ldots\ldots (i)$$

Now we have $f(x) = \sin x$ and $a = \frac{\pi}{2}$, as we are to expand in powers of $\left(x - \frac{\pi}{2}\right)$

$f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$ $f^{iv}(x) = \sin x$ etc.

putting $x = \frac{\pi}{2}$, we get $f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$

$$f'\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0;\ f''\left(\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$$

$$f''\left(\frac{\pi}{2}\right) = -\cos\frac{\pi}{2} = 0;\ f^{iv}\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1 \text{ etc.}$$

From (i), putting $a = \frac{\pi}{2}$ and substituting these values, we get

$$\sin x = 1 + \left(x - \frac{\pi}{2}\right)(0) + \frac{\left(x - \frac{\pi}{2}\right)^2}{\lfloor 2}(-1) + \frac{\left(x - \frac{\pi}{2}\right)^3}{\lfloor 3}(0) + \frac{\left(x - \frac{\pi}{2}\right)^4}{\lfloor 4}(1) + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$= 1 - \frac{1}{\lfloor 2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{\lfloor 4}\left(x - \frac{\pi}{2}\right)^4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ Answer.}$$

## EXPANSION OF FUNCTION OF SEVERAL VARIABLES:

## TAYLOR'S SERIES OF TWO VARIABLES:

If $f(x,y)$ and all its partial derivatives upto the nth order are finite and continuous for all point $(x,y)$ where

$a \le x \le a+h, \quad b \le y \le b+k$

$$\text{Then } f(a+h, b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f + \frac{1}{\lfloor 2}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f + \frac{1}{\lfloor 3}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f + \ldots\ldots\ldots$$

**Proof:** Suppose that $f(x + h, y + k)$ is a function of one variable only, say $x$ where $y$ is assumed as constant. Expanding by Taylor's theorem for one variable, we have

$$f(x + \delta x, y + \delta y) = f(x, y + \delta y) + \delta x\frac{\partial}{\partial x}\,f(x, y + \delta y) + \frac{(\delta x)^2}{\lfloor 2}\frac{\partial^2}{\partial x^2}f(x, y + \delta y) + \ldots\ldots\ldots\ldots$$

Now expanding for y, we get

$$f(x + \delta x, y + \delta y) = \left[f(x,y) + \delta y\frac{\partial}{\partial y}\,f(x,y) + \frac{(\delta y)^2}{\lfloor 2}\cdot\frac{\partial^2}{\partial y^2}f(x,y) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\right]$$

$$+\delta x.\frac{\partial}{\partial x}\left[f(x,y)+\delta y\frac{\partial}{\partial y}f(x,y)+\dots\dots\dots\right]+\frac{(\delta x)^2}{\underline{|2}}\frac{\partial^2}{\partial x^2}\left[f(x,y)+\delta y\frac{\partial}{\partial y}(x,y)+\dots\dots\dots\right]+..$$

$$=\left[f(x,y)+\delta y\frac{\partial}{\partial y}f(x,y)+\frac{(\partial y)^2}{\underline{|2}}\frac{\partial^2}{\partial y^2}f(x,y)+\dots\dots\dots\right]+$$

$$\delta x\left[\frac{\partial}{\partial x}f(x,y)+\delta y\frac{\partial^2}{\partial x\partial y}f(x,y)\right]+\frac{(\partial y)^2}{\underline{|2}}\left[\frac{\partial^2}{\partial x^2}f(x,y)+\dots\dots\dots\right]+\dots\dots\dots$$

$$=f(x,y)+\left[\delta x\frac{\partial}{\partial x}f(x,y)+\delta y\frac{\partial}{\partial y}f(x,y)\right]+\frac{1}{\underline{|2}}\left[(\delta x)^2\frac{\partial^2 f(x,y)}{\partial x^2}+\right.$$

$$\left.2\delta x\,\delta y\frac{\partial^2 f(x,y)}{\partial x\partial y}+(\delta y)^2\frac{\partial^2 f(x,y)}{\partial y^2}\right]+\dots\dots\dots$$

$$\Rightarrow f(a+h,b+k)=f(a,b)+\left[h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial x}\right]+\frac{1}{\underline{|2}}\left[h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\partial y}+k^2\frac{\partial^2 f}{\partial y^2}\right]+\dots\dots\dots$$

$$\Rightarrow f(a+h,b+k)=f(a,b)+\left[h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right]f+\frac{1}{\underline{|2}}\left[h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right]^2 f+\dots\dots\dots\dots$$

on putting a = 0, b =0, h= x, k =y, we get f(x,y) =

$$f(0,0)+\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)+\frac{1}{\underline{|2}}\left(x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 y}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}\right)+\dots\dots\dots\dots\dots\dots$$

**Example 1:** Expand $e^x$ sin y in powers of x and y as for terms of the third degree,

**Solution:** Here f(x,y) = $e^x$ sin y ⇒ f(0,0) = 0

$f_x(x,y) = e^x \sin y \Rightarrow f_x(0,0) = 0$

$f_y(x,y) = e^x \cos y \Rightarrow f_y(0,0) = 1$

$f_{xx}(x,y) = e^x \sin y \Rightarrow f_{xx}(0,0) = 0$

$f_{xy}(x,y) = e^x \cos y \Rightarrow f_{xy}(0,0) =1$

$f_{yy}(x,y) = - e^x \sin y \Rightarrow f_{yy}(0,0) = 0$

$f_{xxx}(x,y) = e^x \sin y \Rightarrow f_{xxx}(0,0) = 0$

$f_{xxy}(x,y) = e^x \cos y \Rightarrow f_{xxy}(0,0) = 1$

$f_{xyy}(x,y) = - e^x \sin y \Rightarrow f_{xyy}(0,0) = 0$

$f_{yyy}(x,y) = - e^x \cos y \Rightarrow f_{yyy}(0,0) = -1$

Then by Taylor's theorem, we have f(x,y)= f(0,0) +

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)f(0,0)+\frac{1}{\underline{|2}}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2 f(0,0)+\frac{1}{\underline{|3}}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2 f(0,0)+\dots\dots\dots\dots$$

$$= f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{x^2}{\underline{|2}} f_{xx}(0,0) + \frac{2xy}{\underline{|2}} f_{xy}(0,0) + \frac{y^2}{\underline{|2}} f_{yy}(0,0) + \frac{1}{\underline{|3}} x^3 f_{xxx}(0,0) +$$

$$\frac{3x^2y}{\underline{|3}} f_{xxy}(0,0) + \frac{3}{\underline{|3}} xy^2 f_{xyy}(0,0) + \frac{1}{\underline{|3}} y^3 f_{yyy}(0,0) + \ldots\ldots\ldots\ldots$$

$$\Rightarrow e^x \sin y = 0 + x(0) + y(1) + \frac{x^2}{2}(0) + xy\ (1) + \frac{y^2}{2}(0)$$

$$+ \frac{x^3}{6}(0) + \frac{3x^2y}{6}(1) + \frac{3xy^2}{6}(0) + \frac{y^3}{6}(-1) + \ldots\ldots\ldots\ldots$$

$$\Rightarrow e^x \sin y = y + xy + \frac{x^2y}{2} - \frac{y^3}{6} + \ldots\ldots\ldots\ldots \text{ Answer}$$

**Example 2:** Expand $e^x \cos y$ near the point $\left(1, \frac{\pi}{4}\right)$ by Taylor's theorem.

(U.P.T.U 2007)

**Solution:** We have

$$f(x+h, y+k) = f(x,y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f + \frac{1}{\underline{|2}}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f + \frac{1}{\underline{|3}}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f + \ldots\ldots\ldots$$

Again $e^x \cos y = f(x,y) = f\left[1 + (x-1), \frac{\pi}{4} + \left(y - \frac{\pi}{4}\right)\right]$

where $h = x-1$ and $k = y - \frac{\pi}{4}$

$e^x \cos y = f(1+h, \frac{\pi}{4} + k)$

$\therefore$ $f(x,y) = e^x \cos y \Rightarrow \left(1, \frac{\pi}{4}\right) = \frac{e}{\sqrt{2}}$

$$\frac{\partial f}{\partial x} = e^x \cos y \Rightarrow \frac{\partial f\left(1, \frac{\pi}{4}\right)}{\partial x} = \frac{e}{\sqrt{2}}$$

$$\frac{\partial f}{\partial y} = -e^x \sin y \Rightarrow \frac{\partial f\left(1, \frac{\pi}{4}\right)}{\partial y} = -\frac{e}{\sqrt{2}}$$

$$\frac{\partial^2 f}{\partial x^2} = e^x \cos y \Rightarrow \frac{\partial^2 f\left(1, \frac{\pi}{4}\right)}{\partial x^2} = \frac{e}{\sqrt{2}}$$

$$\frac{\partial^2 f}{\partial y^2} = -e^x \cos y \Rightarrow \frac{\partial^2 f\left(1,\frac{\pi}{4}\right)}{\partial y^2} = -\frac{e}{\sqrt{2}}$$

$$\frac{\partial^2 f}{\partial x \partial y} = -e^x \sin y \Rightarrow \frac{\partial^2 f\left(1,\frac{\pi}{4}\right)}{\partial x \partial y} = -\frac{e}{\sqrt{2}}$$

Substituting these values in Taylor's Theorem, we obtain

$$e^x \cos y = \frac{e}{\sqrt{2}} + \left[(x-1)\frac{e}{\sqrt{2}} + \left(y-\frac{\pi}{4}\right)\left(-\frac{e}{\sqrt{2}}\right)\right]$$
$$+\frac{1}{\underline{|2}}\left[(x-1)^2\frac{e}{\sqrt{2}} + 2(x-1)\left(y-\frac{\pi}{4}\right)\left(-\frac{e}{\sqrt{2}}\right) + \left(y-\frac{\pi}{4}\right)^2\left(-\frac{e}{\sqrt{2}}\right)\right] + \ldots\ldots\ldots$$

Answer.

**Example 3:** Expand $\tan^{-1}\frac{y}{x}$ in the neighbourhood of (1,1) by Taylor's theorem. Hence compute f(1.1, 0.9).

(U.P.T.U. 2002, 2005)

**Solution:** Here $f(x,y) = \tan^{-1}\frac{y}{x}$ ..................(i)

$a = 1, b = 1$

$\therefore\ f(1,1) = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}\frac{\pi}{4}$ putting x = y =1 in (i)

By Taylor's theorem , we have

$$f(x,y) = f(a,b) + \left[(x-a)\frac{\partial f}{\partial x} + (y-b)\frac{\partial f}{\partial y}\right] + \frac{1}{\underline{|2}}\left[(x-a)^2\frac{\partial^2 f}{\partial x^2} + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y} + (y-b)^2\frac{\partial^2 f}{\partial y^2}\right] + \ldots\ldots$$

Here a= 1, b = 1 so we have

$$f(x,y) = f(1,1) + \left[(x-1)\frac{\partial f}{\partial x} + (y-1)\frac{\partial f}{\partial y}\right] + \frac{1}{\underline{|2}}\left[(x-1)^2\frac{\partial^2 f}{\partial x^2} + 2(x-1)(y-1)\frac{\partial^2 f}{\partial x \partial y} + (y-1)^2\frac{\partial^2 f}{\partial y^2}\right] + \ldots\textbf{(ii)}$$

Now from (i) we have

$$f(x,y) = \tan^{-1}\frac{y}{x} \Rightarrow f(1,1) = \frac{\pi}{4}$$

$$\therefore \frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2} \Rightarrow \left(\frac{\partial f}{\partial x}\right)_{(1,1)} = -\frac{1}{2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2xy}{\left(x^2+y^2\right)^2} \Rightarrow \left(\frac{\partial^2 f}{\partial x^2}\right)_{(1,1)} = \frac{1}{2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{y^2 - x^2}{\left(y^2 + x^2\right)^2} \Rightarrow \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(1,1)} = 0$$

$$\frac{\partial f}{\partial y} = \frac{x}{x^2 + y^2} \quad \Rightarrow \left(\frac{\partial f}{\partial y}\right)_{(1,1)} = \frac{1}{2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-2xy}{\left(x^2 + y^2\right)^2} \quad \Rightarrow \left(\frac{\partial^2 f}{\partial y^2}\right)_{(1,1)} = -\frac{1}{2}$$

Substituting the values of $f(1,1)\left(\frac{\partial f}{\partial x}\right)_{(1,1)}\left(\frac{\partial f}{\partial y}\right)_{(1,1)}, \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(1,1)}$ etc in (ii) we have

$$\tan^{-1}\frac{y}{x} = \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{\lfloor 2}\left[(x-1)^2 . \frac{1}{2} + 2(x-1)(y-1).0 + (y-1)^2\left(-\frac{1}{2}\right)\right] + \dots\dots\dots$$

$$\Rightarrow \tan-1\frac{y}{x} = \frac{\pi}{4} - \left(\frac{x-1}{2}\right) + \left(\frac{y-1}{2}\right) + \frac{1}{\lfloor 2}\left[\frac{(x-1)^2 - (y-1)^2}{2}\right] + \dots\dots\dots\dots\dots\text{(iii)}$$

putting (x-1) = 1.1 - 1 = 0.1, (y-1) = 0.9 -1 = - 0.1, we get

$$f(1.1, 0.9) = \frac{\pi}{4} - \frac{1}{2}\ (0.1) - \frac{1}{2}(-0.1) + \frac{1}{4}\ (0.1)^2 = -\frac{1}{4}\ (-0.1)^2$$

= 0.8862 - 0.1

= 0.7862 Answer.

**Example 4:** Expand $x^y$ in powers of (x-1) and (y-1) upto the third degree terms.

(U.P.T.U 2003)

**Solution:** We have $f(x,y) = x^y$

Here taking a =1 and b =1 we have Taylor's expansion as

$$f(x,y) = f(a,b) + \left[(x-a)\frac{\partial f}{\partial x} + (y-b)\frac{\partial f}{\partial y}\right] + \frac{1}{\lfloor 2}\left[(x-a)^2\frac{\partial^2 f}{\partial x^2} + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y} + (y-b)^2\frac{\partial^2 f}{\partial y^2}\right]$$

$$+\frac{1}{\lfloor 3}\left[(x-a)^3\frac{\partial^3 f}{\partial x^3} + 3(x-a)^2(y-b)\frac{\partial^3 f}{\partial x \partial x \partial y} + 3(x-a)(y-a)^2\frac{\partial^3 f}{\partial x \partial y \partial y} + (y-b)^3\frac{\partial^3 f}{\partial y^3}\right] \quad (1)$$

Now $f(x,y) = x^y \Rightarrow f(1,1) = 1$

$f_x(x,y) = yx^{y-1} \Rightarrow f_x(1,1) = 1$

$f_y(x,y) = x^y \log x \Rightarrow f_y(1,1) = 0$

$f_{xx}(x,y) = y(y-1)x^{y-2} \Rightarrow f_{xx}(1,1) = 0$

$f_{xy}(x,y) = x^{y-1} + yx^{y-1}\log x \Rightarrow f_{xy}(0,0) = 1$

$f_{yy}(x,y) = x^y(\log x)^2 \Rightarrow f_{yy}(1,1) = 0$

$f_{xxx}(x,y) = y(y-1)(y-2)x^{y-3} \Rightarrow f_{xxx}(1,1) = 0$

$f_{xxy}(x,y) = (y-1)x^{y-2} + y(y-1)\ x^{y-2}\log x + yx^{y-2} \Rightarrow f_{xxy}(1,1) = 1$

$f_{xyy}(x,y) = yx^{y-1}(\log x)^2 + x^y\,\frac{2\log x}{x} = yx^{y-1}(\log x)^2+2x^{y-1}\log x \Rightarrow f_{xyy}(1,1) = 0$

$f_{yyy}(x,y) = x^y(\log x)^3 \Rightarrow f_{yyy}(0,0) = 0$

Substituting these values in the Taylor's expansion, we get

$$x^y = 1+(x-1)+0+\frac{1}{\underline{|2}}[0+2(x-1)(y-1)+0]+\frac{1}{\underline{|3}}[0+3(x-1)^2(y-1)+0+0]$$

$\Rightarrow x^y = 1+(x-1)+(x-1)(y-1)+\frac{1}{2}(x-1)^2(y-1)$ Answer.

**Example 5:** Find the first six terms of the expansion of the function $e^x\log(1+y)$ in a Taylor's series in the neighourhood of the point (0,0)

**Solution:** Here $f(x,y) = e^x\log(1+y) \Rightarrow f(0,0) = 0$

$f_x(x,y) = e^x\log(1+y) \Rightarrow f_x(0,0) = 0$

$f_y(x,y) = \frac{e^x}{1+y} \Rightarrow f_y(0,0) = 1$

$f_{xx}(x,y) = e^x\log(1+y) \Rightarrow f_{xx}(0,0) = 0$

$f_{xy}(x,y) = \frac{e^x}{1+y} \Rightarrow f_{xy}(0,0) = 1$

$f_{yy}(x,y) = \frac{-e^x}{(1+y)^2} \Rightarrow f_{yy}(0,0) = -1$

$f_{xxx}(x,y) = e^x\log(1+y) \Rightarrow f_{xxx}(0,0) = 0$

$f_{xxy}(x,y) = \frac{e^x}{1+y} \Rightarrow f_{xxy}(0,0) = 1$

$f_{xyy}(x,y) = -\frac{e^x}{(1+y)^2} \Rightarrow f_{xyy}(0,0) = -1$

$f_{yyy}(x,y) = \frac{2e^x}{(1+y)^3} \Rightarrow f_{yyy}(0,0) = 2$

we know that

$$f(x,y) = f(0,0)+[x\,f_x(0,0)+y\,f_y(0,0)]+\frac{1}{2}[x^2 f_{xx}(0,0,)+2xy\,f_{xy}(0,0)+y^2 f_{yy}(0,0)]+$$
$$\frac{1}{\underline{|3}}[x^3 f_{xxx}(0,0)+3x^2y\,f_{xxy}(0,0)+3xy^2 f_{xyy}(0,0)+y^3 f_{yyy}(0,0)]+\ldots\ldots\ldots\ldots$$

$$\therefore e^x\log(1+y) = 0+[x.0+y.1]+\frac{1}{2}[x^2.\,0+2xy.1+y^2(-1)]+\frac{1}{6}[x^3.\,0+3x^2y.1+3xy^2(-1)+y^3.2]+\ldots\ldots\ldots\ldots\ldots$$

$e^x\log(1+y) = y+xy-\frac{1}{2}y^2+\frac{1}{2}x^2y-\frac{1}{2}xy^2+\frac{1}{3}y^3+\ldots\ldots\ldots\ldots$ Answer.

**EXERCISE**

1. Apply Taylor's theorem to find the expansion of log sin(x + h).

Ans. log sin (x + h) = log sin x + h cot x $-\frac{h^2}{2}$ cosec$^2$ x + $\frac{h^3}{3}$ cosec$^2$ cot x + .........

2. Expand $2x^3 + 7x^2 + x - 1$ in powers of (x -2).

(B.P.S.C. 2007)

Ans. $45 + 53(x-2) + 19(x-2)^2 + 2(x-3)^2$

3. Expand $x^2y + 3y - 2$ in powers of (x–1) and (y + 2) using Taylor's theorem.

(U.P.P.C.S. 1992; P.T.U. 2006)

Ans. $-10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1) + (y+2) + (x-1)^2(y+2)$

4. Expand $\frac{(x+h)(y+k)}{(x+h)+(y+k)}$ in powers of h and k upto and inclusive of the 2nd degree terms.

Ans. $\frac{xy}{x+y} + \frac{hy^2 + kx^2}{(x+y)^2} - \frac{h^2y^2}{(x+y)^3} + \frac{2hkxy}{(x+y)^3} - \frac{k^2x^2}{(x+y)^3} + \ldots\ldots\ldots\ldots$

5. If f(x,y) = $\tan^{-1}$ (xy), compute an approximate value of f(0.9, -1.2)

Ans. - 0.823

6. Find the expansion for cos x cos y in powers of x, y up to fourth order terms.

Ans. cos x cos y = $1 - \frac{x^2}{2} - \frac{y^2}{2} + \frac{x^4}{24} + \frac{x^2y^2}{4} + \frac{y^4}{24} + \ldots\ldots\ldots$

7. Expand sin (xy) about the point $\left(1, \frac{\pi}{2}\right)$ upto and including second degree terms using Taylor's series.

Ans. $1 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{2}(x-1)\left(y - \frac{\pi}{2}\right) - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2$

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# UNIT - 2
# Differential Calculus - II

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# Chapter 5
# Jacobians

## JACOBIANS

**Jacobians :** If u and v are functions of the two independent variables x and y, then the determinant.

(U.P.P.C.S. 1990)

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is called the Jacobian of u, v with respect to x,y and is written as

$$\frac{\partial(u,v)}{\partial(x,y)} \text{ or } J(u,v) \text{ or } J\left(\frac{u,v}{x,y}\right)$$

Similarly if u, v and w be the functions of three independent variables x,y and z, then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

it is written J(u,v,w) or $J\left(\frac{u,v,w}{x,y,z}\right)$

**Properties of Jacobians**

**(1) First Property:**

If u and v are functions of x and y, then

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1 \qquad \text{or } JJ' = 1$$

(U.P.T.U. 2005)

Where $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J' = \frac{\partial(x,y)}{\partial(u,v)}$

**Proof:** Let u= u(x,y) and v = v(x,y), so that u and v are functionus of x and y

$$\text{Now } \frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y}\\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y}\end{vmatrix}\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}$$

on interchanging the rows and column of second determinant

$$=\begin{vmatrix}\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y}\\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y}\end{vmatrix}\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u}\\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v}\end{vmatrix}$$

$$=\begin{vmatrix}\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial x}{\partial u}+\dfrac{\partial u}{\partial y}\cdot\dfrac{\partial y}{\partial u} & \dfrac{\partial u}{\partial x}\cdot\dfrac{\partial x}{\partial v}+\dfrac{\partial u}{\partial y}\cdot\dfrac{\partial y}{\partial v}\\ \dfrac{\partial v}{\partial x}\cdot\dfrac{\partial x}{\partial u}+\dfrac{\partial v}{\partial y}\cdot\dfrac{\partial y}{\partial u} & \dfrac{\partial v}{\partial x}\cdot\dfrac{\partial x}{\partial v}+\dfrac{\partial v}{\partial y}\cdot\dfrac{\partial y}{\partial v}\end{vmatrix} \qquad \text{(i)}$$

Now differentiating u = u(x,y) and v =v(x,y) partially with respect to u and v, we get

$$\left.\begin{aligned}
\frac{\partial u}{\partial u}&=1=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u}\\
\frac{\partial u}{\partial v}&=0=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v}\\
\frac{\partial v}{\partial v}&=1=\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v}\\
\frac{\partial v}{\partial u}&=0=\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u}
\end{aligned}\right\} \qquad \text{(ii)}$$

on making substitutions from (ii) in (i) we get

$$\frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}=1$$

or JJ' = 1 Proved.

**(2) Second Property**

If u, v are the functions of r,s where r,s are functions of x,y then

$$\frac{\partial(u,v)}{\partial(x,y)}=\frac{\partial(u,v)}{\partial(r,s)}\cdot\frac{\partial(r,s)}{\partial(x,y)}$$

**Proof:** $$\frac{\partial(u,v)}{\partial(r,s)}.\frac{\partial(r,s)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial r} & \frac{\partial u}{\partial s}\\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s}\end{vmatrix}\begin{vmatrix}\frac{\partial r}{\partial x} & \frac{\partial r}{\partial y}\\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y}\end{vmatrix}$$

on interchanging the columns and rows in second determinant

$$=\begin{vmatrix}\frac{\partial u}{\partial r} & \frac{\partial u}{\partial s}\\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s}\end{vmatrix}\begin{vmatrix}\frac{\partial r}{\partial x} & \frac{\partial s}{\partial x}\\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{\partial u}{\partial r}.\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}.\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}.\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}.\frac{\partial s}{\partial y}\\ \frac{\partial v}{\partial r}.\frac{\partial r}{\partial x}+\frac{\partial v}{\partial s}.\frac{\partial s}{\partial x} & \frac{\partial v}{\partial r}.\frac{\partial r}{\partial y}+\frac{\partial v}{\partial s}.\frac{\partial s}{\partial y}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$$

$$=\frac{\partial(u,v)}{\partial(x,y)}$$ Proved

**Note -** Similarly we can prove that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\frac{\partial(u,v,w)}{\partial(r,s,t)}.\frac{\partial(r,s,t)}{\partial(x,y,z)}$$

**(3) Third Property**

If functions u,v,w of three independent variables x,y,z are not independent, then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=0$$

**Proof:** As u, v, w are independent, then f (u, v, w) =0 (i)

Differentiating (i) w.r.t x,y, z we get

$$\frac{\partial f}{\partial u}.\frac{\partial u}{\partial x}+\frac{\partial f}{\partial y}.\frac{\partial v}{\partial x}+\frac{\partial f}{\partial w}.\frac{\partial w}{\partial x}=0 \qquad \text{(ii)}$$

$$\frac{\partial f}{\partial u}.\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}.\frac{\partial v}{\partial y}+\frac{\partial f}{\partial w}.\frac{\partial w}{\partial y}=0 \qquad \text{(iii)}$$

$$\frac{\partial f}{\partial u}.\frac{\partial u}{\partial z}+\frac{\partial f}{\partial v}.\frac{\partial v}{\partial z}+\frac{\partial f}{\partial w}.\frac{\partial w}{\partial z}=0 \qquad \text{(iv)}$$

Eliminating $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial f}{\partial w}$ from (ii), (iii) and (iv), we have

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{vmatrix} = 0$$

on interchanging rows and columns, we get

$$\Rightarrow \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = 0$$

$$\Rightarrow \frac{\partial(u,v,w)}{\partial(x,y,z)} = 0 \qquad \text{Proved}$$

**Example 1:** If $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$, Show that the Jacobian of $y_1$, $y_2$, $y_3$ with respect to $x_1$, $x_2$, $x_3$ is 4.

(U.P.P.C.S. 1991, U.P.T.U. 2002, 2004)

**Solution:** Here given

$$y_1 = \frac{x_2 x_3}{x_1}, \quad y_2 = \frac{x_3 x_1}{x_2}, \quad y_3 = \frac{x_1 x_2}{x_3}$$

$$\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{x_2x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_3x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_2} & -\frac{x_1x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2x_2^2x_3^2} \begin{vmatrix} -x_2x_3 & x_3x_1 & x_1x_2 \\ x_2x_3 & -x_3x_1 & x_1x_2 \\ x_2x_3 & x_3x_1 & -x_1x_2 \end{vmatrix}$$

$$= \frac{x_1^2x_2^2x_3^2}{x_1^2x_2^2x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$= -1\,(1-1) -1(-1-1) + 1(1+1)$

$= 0 + 2 + 2$

$= 4$ Proved

**Example 2:** If $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$, show that (U.P.T.U. 2001)

$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2 \sin\theta$ and find $\frac{\partial(r,\theta,\phi)}{\partial(x,y,z)}$

(U.P.T.U. 2001)

**Solution:**

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\theta \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta\left\{\cos\theta\begin{vmatrix} \cos\theta\cos\phi & -\sin\phi \\ \cos\theta\sin\phi & \cos\phi \end{vmatrix} + \sin\theta\begin{vmatrix} \sin\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\phi \end{vmatrix}\right\}$$

$= r^2 \sin\theta\ [\cos\theta\ (\cos\theta \cos^2\phi + \cos\theta \sin^2\phi) + \sin\theta\ (\sin\theta \cos^2\phi + \sin\theta \sin^2\phi)]$

$= r^2 \sin\theta\ (\cos^2\theta + \sin^2\theta)$

$= r^2 \sin\theta$

$\therefore \dfrac{\partial(r,\theta,\phi)}{\partial(x,y,z)} = \dfrac{1}{r^2\sin\theta}$ (using first property) Answer.

**Example 3:** If $u = x\,y\,z$, $v = x^2 + y^2 + z^2$, $w = x + y + z$, find $J = \dfrac{\partial(x,y,z)}{\partial(u,v,w)}$.

(U.P.T.U. 2002, 03)

**Solution:** Since u, v, w are explicitly given, so first we evaluate $J' = \dfrac{\partial(u,v,w)}{\partial(x,y,z)}$.

$$\text{Now } J' = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & zx & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$= yz\ (2y - 2z) - zx\ (2x - 2z) + xy\ (2x - 2y)$

$= 2[yz\ (y - z) - zx\ (x - z) + xy\ (x - y)]$

$=2(x^2y - x^2z - xy^2 + xz^2 + y^2z - yz^2)$

$= 2[x^2 (y - z) -x\ (y^2 - z^2) + yz\ (y - z)]$

$= 2(y - z)\ [\ x^2 - x\ (y + z) + yz]$

$= 2(y - z)\ [y(z - x) -\ x\ (z - x)]$

$= 2(y - z)\ (z -x)\ (y - x)$

$= -2(x - y)\ (y - z)\ (z - x)$

Hence by $JJ' = 1$, we have

$J = \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = -\dfrac{1}{2(x-y)(y-z)(z-x)}$ Answer.

**Example 4:** If $u = x + y + z$, $uv = y + z$, $u\,v\,w = z$, Evaluate $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

(I.A.S. 2005, U.P.P.C.S. 1990, U.P.T.U. 2003)

**Solution:**

$x = u - uv = u\,(1 - v)$

$y = uv - uvw = uv\,(1 - w)$

$z = uvw$

$$\therefore \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v(1-w) & u(1-w) & -uw \\ vw & uw & uv \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 + (R_2 + R_3)$

$= u^2v\,(1 - w) + u^2\,vw$

$= u^2v$ Answer.

**Example 5:** If u, v, w are the roots of the equation $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$, in $\lambda$, find $\dfrac{\partial(u,u,w)}{\partial(x,y,z)}$.

(I.A.S. 2002, 2004; U.P.P.C.S. 1999; B.P.S.C. 2007; U.P.T.U. 2002)

**Solution:** Given

$(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$

$\Rightarrow 3\lambda^3 - 3(x + y + z)\lambda^2 + 3(x^2 + y^2 + z^2)\lambda - (x^3 + y^3 + z^3) = 0$

sum of the roots $= u + v + w = x + y + z$ (i)

product of the roots $= uv + vw + wu = x^2 + y^2 + z^2$ (ii)

$uvw = \frac{1}{3}(x^3 + y^3 + z^3)$ (iii)

Equation (i), (ii) and (iii) can be written as

$f_1 = u + v + \omega - x - y - z$

$f_2 = uv + vw + wu - x^2 - y^2 - z^2$

$f_3 = uvw - \frac{1}{3}\ (x^3 + y^3 + x^3)$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -x^2 & -y^2 & -z^2 \end{vmatrix}$$

$$= (-1)(-2)(-1)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= -2\begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ x^2-y^2 & y^2-z^2 & z^2 \end{vmatrix} \quad \begin{array}{l} \text{Applying} \\ C_1 \to C_1 - C_2 \\ C_1 \to C_2 - C_3 \end{array}$$

$$= -2(x-y)(y-z)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & z \\ x+y & y+z & z^2 \end{vmatrix}$$

$= -2\ (x - y)\ (y - z)\ (y + z\ - x - y)$

$= -2\ (x - y)\ (y - z)\ (z - x)$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ v+w & u+w & u+v \\ vw & wu & uv \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ v-u & w-u & u+v \\ w(v-u) & u(w-v) & uv \end{vmatrix}$$

$$= (v-u)(w-v) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & u+v \\ w & u & uv \end{vmatrix}$$

$=(v - u) (w - v) (u - w)$

$= - (u - v) (v - w) (w - u)$

Therefore $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{\dfrac{\partial(f_1,f_2,f_3)}{-\partial(x,y,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}$$

$$= -\frac{-2(x-y)(y-z)(z-x)}{-(u-v)(v-w)(w-u)}$$

$$= -\frac{2(x-y)(y-z)(z-x)}{(u-v)((v-w)(w-u))}$$ Answer.

**EXERCISE**

1. If $x = r\cos\theta$, $y = r\sin\theta$, evaluate $\frac{\partial(x,y)}{\partial(r,\theta)}$ and $\frac{\partial(r,\theta)}{\partial(x,y)}$

Ans. $r, \frac{1}{r}$

2. If $x = uv$, $y = \frac{u+v}{u-v}$, find $\frac{\partial(u,v)}{\partial(x,y)}$

Ans. $\frac{(u-v)^2}{4uv}$

3. If $u^3 + v^3 = x + y$, $u^2 + v^2 = x^3 + y^3$, prove that $\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{2}\frac{(y^2-x^2)}{uv(u-v)}$

4. If $u = xy + yz + zx$, $v = x^2 + y^2 + z^2$ and $w = x + y + z$, determine whether there is a functional relationship between u,v,w and if so , find it

Ans. $w^2 - v - 2u = 0$, $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

5. If u,v,w are the roots of the equation in λ and $\frac{x}{a+\lambda} + \frac{y}{b+\lambda} + \frac{z}{c+\lambda} = 1$

Then find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

(I.A.S. 2005)

Ans. $\frac{(u-v)(v-w)(w-u)}{(a-b)(b-c)(c-a)}$

6. If u,v,w are the roots of the equation

$(x - a)^3 + (x - b)^3 + (x - c)^3 = 0$ then find $\frac{\partial(u,v,w)}{\partial(a,b,c)}$ (U.P.T.U. 2002)

Ans. $-\frac{2(a-b)(b-c)(c-a)}{(u-v)(v-w)(w-u)}$

# Chapter 6
# Approximation of Errors

Let $z = f(x,y)$ (i)

If $\delta x$, $\delta y$ are small increments in x and y respectively and $\delta z$, the corresponding increment in z, then

$z + \delta z = f(x + \delta x, y + \delta y)$ (ii)

Subtracting (i) from (ii), we get

$\delta z = f(x + \delta x, y + \delta y) - f(x,y)$

$= f(x,y) + \delta x\frac{\partial f}{\partial x} + \delta y\,\frac{\partial f}{\partial y} + \ldots\ldots\ldots\ldots - f(x,y)$

$= \delta x\frac{\partial f}{\partial x} + \delta y\frac{\partial f}{\partial y}$ (approximately)

As neglecting higher powers of $\delta x$, $\delta y$

$\therefore \delta z = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$ (approximately)

$\Rightarrow$ If $\delta x$ and $\delta y$ are small changes (or errors) in x and y respectively, then an approximate change (or error) in z is $\delta z$.

Replacing $\delta x$, $\delta y$, $\delta z$ by dx, dy, dz respectively, we have $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

**Note 1.** If $\delta x$ is the error in x, then relative error $= \frac{\delta x}{x}$

percentage error $= \frac{\delta x}{x} \times 100$

**Note 2.** $f'(x)\,dx$ is called the differential of $f(x)$

**Example 1** The time T of a complete oscillation of a simple pendulum of length L is governed by the equation $T = 2\pi\sqrt{\frac{L}{g}}$,

Find the maximum error in T due to possible errors upto 1% in L and 2% in g.

(U.P.T.U. 2004, 2009)

**Solution :** We have $T = 2\pi\sqrt{\frac{L}{g}}$

$\log T = \log 2\pi + \frac{1}{2}\log_e L - \frac{1}{2}\log_e g$

Differentiating

$$\frac{dT}{T} = 0 + \frac{1}{2}\frac{dL}{L} - \frac{1}{2}\frac{dg}{g}$$

$$\Rightarrow \left(\frac{dT}{T}\right) \times 100 = \frac{1}{2}\left[\left(\frac{dL}{L}\right) \times 100 - \left(\frac{dg}{g}\right) \times 100\right]$$

But $\frac{dL}{L} \times 100 = 1, \frac{dg}{g} \times 100 = 2$

so

$$\frac{dT}{T} \times 100 = \frac{1}{2}[1 \pm 2] = \frac{3}{2}$$

Maximum error in T = 1.5% Answer.

**Example 2:** The power dissipated in a resistor is given by $P = \frac{E^2}{R}$. Find by using calculus the approximate percentage change in P when E is increased by 3% and R is decreased by 2%.

**Solution:** Here given $P = \frac{E^2}{R}$

Taking logarithm we have

$\log P = 2 \log E - \log R$

on differentiating, we get

$$\frac{\delta P}{P} = \frac{2}{E}\delta E - \frac{\delta R}{R}$$

$$\text{or } 100\frac{\delta P}{P} = 2 \times \frac{100\delta E}{E} - \frac{100\delta R}{R}$$

$$\text{or } 100\frac{\delta P}{P} = 2(3) - (-2)$$

$= 8$

Percentage change in P = 8% Answer.

**Example 3:** In estimating the number of bricks in a pile which is measured to be (5m × 10m × 5m), count of bricks is taken as 100 bricks per $m^3$. Find the error in the cost when the tape is stretched 2% beyond its standard length. The cost of bricks is Rs. 2,000 per thousand bricks.

(U.P.T.U. 2000, 2004)

**Solution:** We have volume $V = xyz$

Taking log of both sides, we have

$\log V = \log x + \log y + \log z$

Differentiating, we get

$$\frac{\delta V}{V}=\frac{\delta x}{x}+\frac{\delta y}{y}+\frac{\delta z}{z}$$

$$100\frac{\delta V}{V}=100\frac{\delta x}{x}+100\frac{\delta y}{y}+100\frac{\delta z}{z}$$

= 2 + 2 + 2 (As given)

= 6

$$\Rightarrow \delta V=6\frac{V}{100}=6\frac{(5\times 10\times 5)}{100}$$

= 15 cubic metre

Number of bricks in $\delta V = 15\times100$

= 1500

Error in cost $\dfrac{1500\times 2000}{1000}$

= 3000

This error in cost, a loss to the seller of bricks = Rs. 3000. Answer.

**Example 4:** What error in the common logarithm of a number will be produced by an error of 1% in the number.

**Solution:** Consider x as any number and, let

$y=\log_{10}x$

Then $\delta y=\frac{1}{x}\log_{10}{}^{e}\ \delta x$

$$=\frac{\delta x}{x}\log_{10}e$$

$$=\left(\frac{\delta x}{x}\times 100\right)\left(\frac{1}{100}\log_{10}e\right)$$

$=\frac{1}{100}\log_{10}e \quad \therefore$ as given $\frac{\delta x}{x}\times100=1$

$$=\frac{0.43429}{100}$$

= 0.0043429

which is the required error. Answer

**Example 5:** A balloon is in the form of right circular cylinder of radius 1.5m and length 4m and is surmounted by hemispherical ends. If the radius is increased by 0.01 m and length by 0.05m, find the percentage change in the volume of balloon. (U.P.T.U. 2002, 2005)

**Solution:** Here given,

radius of the cylinder (r) = 1.5m

length of the cylinder (h) = 4m

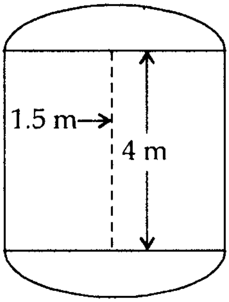
$\delta r = 0.01$m, $\delta h = 0.05$m

Let V be the volume of the balloon then.

V= volume of the circular cylinder + 2 (volume of the hemisphere)

$= \pi r^2 h + 2\left(\frac{2}{3}\pi r^3\right)$

$= \pi r^2 h + \frac{4}{3}\pi r^3$



Now $\delta V = \pi 2r\delta rh, h + \pi r^2 \delta h + \frac{4}{3}\pi.3r^2 \delta r$

$= \pi r[2h\delta r + r\delta h + 4r\delta r]$

$$\Rightarrow \frac{\delta V}{V} = \frac{\pi r\left[2(h+2r)\delta r + r\delta h\right]}{\pi r^2 h + \frac{4}{3}\pi r^3}$$

$$= \frac{2(h+2r)\delta r + r\delta h}{rh + \frac{4}{3}r^2}$$

$$= \frac{2(4+3)(0.01)+(1.5)(0.05)}{(1.5\times 4)+\frac{4}{3}(1.5)^2}$$

$$= \frac{0.14+0.075}{6+3} = \frac{0.215}{9}$$

$\therefore \frac{\delta V}{V}\times 100 = \frac{0.215}{9}\times 100 = \frac{21.5}{9} = 2.389\%$ Answer.

**Example 6:** If the base radius and height of a cone are measured as 4 and 8 inches with a possible error of 0.04 and 0.08 inches respectively, calculate the percentage (%) error in calculating volume of the cone.

[U.P.T.U. (C.O.) 2003]

**Solution:**

Volume $V = \frac{1}{3}\pi r^2 h$

Taking log,

$\log V = \log \frac{1}{3} + \log \pi + 2 \log r + \log h$

Differentiating, we get

$$\frac{\delta V}{V} = 2\frac{\delta r}{r} + \frac{\delta h}{h}$$

$$\Rightarrow \frac{\delta V}{V} = 2\left(\frac{0.04}{4}\right) + \left(\frac{0.08}{8}\right)$$

$= 0.03$

∴ Percentage (%) error in volume

$= 0.03 \times 100$

$= 3\%$ Answer.

**Example 7:** Find the percentage error in the area of an ellipse when an error of +1 percent is made in measuring the major and minor axes.

**Solution:** If x and y are semi-major and semi-minor axes of ellipse. Then its area A is given by

$A = \pi xy$

Taking log,

$\log A = \log \pi + \log x + \log y$

Differentiating, we get

$$\frac{\delta A}{A} = 0 + \frac{\delta x}{x} + \frac{\delta y}{y}$$

$$\text{or } \frac{\delta A}{A} \times 100 = \frac{\delta x}{x} \times 100 + \frac{\delta y}{y} \times 100$$

$= 1+1$

$= 2$

∴ Error in the area = 2% Answer.

**Example 8:** Find the possible percentage error in computing the parallel resistance r of three resistance $r_1$, $r_2$, $r_3$ from the formula.

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

If $r_1$, $r_2$, $r_3$ are each in error by + 1.2%

(U.P.T.U. 1999)

**Solution:** we have

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \qquad \text{(i)}$$

Differentiating, we get

$$-\frac{1}{r^2}\delta r = -\frac{1}{r_1^2}\delta r_1 - \frac{1}{r_2^2}\delta r_2 - \frac{1}{r_3^2}\delta r_3$$

$$\Rightarrow \frac{1}{r}\left(\frac{\delta r}{r}\times 100\right) = \frac{1}{r_1}\left(\frac{\delta r_1}{r_1}\times 100\right) + \frac{1}{r_2}\left(\frac{\delta r_2}{r_2}\times 100\right) + \frac{1}{r_3}\left(\frac{\delta r_3}{r_3}\times 100\right)$$

$$= \frac{1}{r_1}(1.2) + \frac{1}{r_2}(1.2) + \frac{1}{r_3}(1.2)$$

$$= 1.2\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right)$$

$$= 1.2\left(\frac{1}{r}\right) \text{ from (i)}$$

$$\Rightarrow \frac{\delta r}{r}\times 100 = 1.2$$

Hence % error in r = 1.2 Answer.

**Example 9:** The angles of a triangle are calculated from the sides a,b,c. If small changes δa, δb and δc area made in the sides, find δA, δB and δC approximately, where Δ is the area of the triangle and A, B, C are angles opposite to sides a, b, c respectively. Also show that $\delta A + \delta B + \delta C = 0$

(U.P.T.U. 2002)

**Solution:** We know that

$a^2 = b^2 + c^2 - 2abc \cos A$ (i)

Differentiating eqn (i), we get

$2a\delta a = 2b\delta b + 2c\delta c - 2b\delta c \cos A - 2c\delta b \cos A + 2bc \sin A\, \delta A$

or $bc \sin A\, \delta A = a\delta a - (b - c\cos A)\,\delta b - (c - b\cos A)\,\delta c$

or $2\Delta\delta A = a\delta a - (a\cos C + c\cos A - c\cos A)\,\delta b - (a\cos B + b\cos A - b\cos A)\,\delta c$

or $\delta A = \frac{a}{2\Delta}(\delta a - \delta b\cos C - \delta c\cos B)$ (ii)

similarly, also by symmetry, we have

$\delta B = \frac{b}{2\Delta}(\delta b - \delta c\cos A - \delta a\cos C)$ (iii)

and $\delta C = \frac{c}{2\Delta}(\delta c - \delta a\cos B - \delta b\cos A)$ (iv)

Adding equations (ii), (iii) and (iv) we get

$\delta A + \delta B + \delta C = \frac{1}{2\Delta}[(a - b\cos C - c\cos B)\,\delta a + (b - c\cos A - a\cos C)\,\delta b + (c - a\cos B - b\cos A)\delta c]$

$= \frac{1}{2\Delta}[(a - a)\,\delta a + (b - b)\,\delta b + (c - c)\,\delta c]$

$= \frac{1}{2\Delta}(0+0+0)$

$= 0$

Thus, $\delta A + \delta B + \delta C = 0$ Hence Proved.

**Example 10:** Two sides a, b of a triangle and included angle C are measured, show that the error δc in the computed length of third side c due to a small error in the angle C is given by

$\delta C\,(a \sin B) = \delta c$

**Solution:** we know that

$\cos C = \frac{a^2 + b^2 + c^2}{2ab}$, where a, b are fixed while c and C vary

$$\therefore \sin C\, \delta C = \frac{-2c\delta c}{2ab}$$

$$\Rightarrow \delta c = \frac{ab}{c} \sin C\, \delta C \qquad \text{(i)}$$

Again by sine formula, we have

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \sin B = \frac{b \sin C}{c}$$

$\Rightarrow \delta c = a \sin B\, \delta C$ Proved.

**Example 11:** If Δ be the area of a triangle, prove that the error in Δ resulting from a small error in c is given by

$$\delta\Delta = \frac{\Delta}{4}\left[\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-b} - \frac{1}{s-c}\right]\delta c$$ (U.P.T.U. C.O. 2006-07)

**Solution:** we know that

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\Rightarrow \log \Delta = \frac{1}{2}[\log s + \log (s - a) + \log (s - b) + \log (s - c)]$$

Differentiating w.r.t.c, we get

$$\frac{1}{\Delta}\frac{\delta\Delta}{\delta c} = \frac{1}{2}\left[\frac{1}{s}\frac{\delta s}{\delta c} + \frac{1}{(s-a)}\frac{\delta(s-a)}{\delta c} + \frac{1}{(s-b)}\frac{\delta(s-b)}{\delta c} + \frac{1}{(s-c)}\frac{\delta(s-c)}{\delta c}\right] \qquad \text{(i)}$$

Now $s = \frac{1}{2}(a + b + c)$

$$\therefore \frac{\delta s}{\delta c} = \frac{1}{2}$$

Also $s - c = \frac{1}{2}(a + b - c)$

$$\therefore \frac{\delta(s-c)}{\delta c} = -\frac{1}{2}$$

similarly, $s - a = \frac{1}{2}(b + c - a) \Rightarrow \frac{\delta(s-a)}{\delta c} = \frac{1}{2}$

and $s - b = \frac{1}{2}(a + b - c) \Rightarrow \frac{\delta(s-a)}{\delta c} = \frac{1}{2}$

From (i)

$$\frac{1}{\Delta}\frac{\delta\Delta}{\delta c} = \frac{1}{2}\left[\frac{1}{s}\left(\frac{1}{2}\right) + \frac{1}{s-a}\left(\frac{1}{2}\right) + \frac{1}{s-b}\left(\frac{1}{2}\right) + \frac{1}{(s-c)}\left(-\frac{1}{2}\right)\right]$$

$$\Rightarrow \delta\Delta = \frac{\Delta}{4}\left[\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-b} - \frac{1}{s-c}\right]\delta c \qquad \text{Proved.}$$

**Example 12:** If the kinetic energy T is given by $T = \frac{1}{2}mv^2$ find approximate the change in T as the mass m change from 49 to 49.5 and the velocity v changes from 1600 to 1590.

**Solution:** Here given

$$T = \frac{1}{2}mv^2$$

$$\therefore \delta T = \frac{1}{2}\{(\delta m)\, v^2 + m\,(2v\delta v)\}$$

$$\text{or } \delta T = \frac{1}{2}v^2\,\delta m + mv\,\delta v \qquad \text{(i)}$$

It is given that m changes from 49 to 49.5

$\therefore \delta m = 0.5$ (ii)

Also, v changes from 1600 to 1590

$\therefore \delta v = -10$ (iii)

Hence, from (i)

$$\delta T = \frac{1}{2}(1600)^2(0.5) + 49(1600)(-10)$$

$= -144000$

Thus, T decreases by 144000 units. Answer.

**Example 13:** Find approximate value of $[(0.98)^2 + (2.01)^2 + (1.94)^2]^{1/2}$

**Solution:** Let $f(x, y, z) = (x^2 + y^2 + z^2)^{1/2}$ (i)

Taking $x = 1$, $y = 2$ and $z = 2$ so that $dx = -0.02$, $dy = 0.01$ and $dx = -0.06$ from (i)

$$\frac{\partial f}{\partial x} = x\,(x^2 + y^2 + z^2)^{-1/2},\ \frac{\partial f}{\partial y} = y\,(x^2 + y^2 + z^2)^{-1/2},\ \frac{\partial f}{\partial z} = z\,(x^2 + y^2 + z^2)^{-1/2}$$

Now $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$ (by total differentiation)

$= (x^2 + y^2 + z^2)^{-1/2} (xdx + ydy + zdz)$

$= \frac{1}{3}(-0.02 + 0.02 - 0.12)$

$= -0.04$

$\therefore [(0.98)^2 + (2.01)^2 + (1.94)2]^{1/2} = f(1,2,2) + df$

$= 3 + (-0.04)$

$= 2.96$ Answer.

**Example 14:** In determining the specific gravity by the formula $S = \frac{A}{A - w}$ where A is the weight in air, w is the weight in water, A can be read within 0.01 gm and w within 0.02 gm. Find approximately the maximum error in S if the readings are A = 1.1 gm, w = 0.6 gm. Find also the maximum relative error.

**Solution :** Given

$$S = \frac{A}{A - w}$$

Differentiating above, we have

$$\delta S = \frac{(A - w)\delta A - A(\delta A - \delta w)}{(A - w)^2} \qquad \text{(i)}$$

Maximum error in S can be obtained if we take $\delta A = -0.01$ and $\delta w = 0.02$

$\therefore$ From (i)

$$\delta S = \frac{(1.1 - 0.6)(-0.01) - (1.1)(-0.01 - 0.02)}{(1.1 - 0.6)^2}$$

$= 0.112$

Maximum relative error in

$$S = \frac{(\delta S)\max}{S} = \frac{0.112}{\left(\frac{1.1}{1.1 - 0.6}\right)}$$

$= 0.05091$ Answer.

**EXERCISE**

1. The deflection at the centre of a rod, of length $l$ and diameter d supported at its ends and loaded at the centre with a weight w varies as $wl^3 d^{-4}$. What is the percentage increase in the deflection corresponding to the percentage increases in w, l and d of 3%, 2% and 1% respectively.

Ans. 5%

2. The work that must be done to propel a ship of displacement D for a distance S in time t is proportional to $\frac{S^2D^{2/3}}{t^2}$. Find approximately the increase of work necessary when the displacement is increased by 1%, the time diminished by 1% and the distance diminished by 2%.

Ans. $-\frac{4}{3}\%$

3. The indicated horse power of an engine is calculated from the formula $I=\frac{PLAN}{32000}$, where $A=\frac{\pi d^2}{4}$. Assuming that errors of r percent may have been made in measuring P, L, N and d, find the greatest possible error in 1%.

Ans. 5r%.

4. In estimating the cost of a pile of bricks measured as 6m × 50m × 4m, the tape is stretched 1% beyond the standard length. If the count is 12 bricks in $1m^3$ and bricks cost Rs. 100 per 1,000 find the approximate error in the cost.

(U.P.T.U. 2005)

Ans. Rs 43.20

5. If the sides and angles of a triangle ABC vary in such a way that its circum radius remains constant, Prove that $\frac{\delta a}{\cos A}+\frac{\delta b}{\cos B}+\frac{\delta c}{\cos C}=0$

6. Compute an approximate value of $(1.04)^{3.01}$

Ans. 1.12

7. Evaluate $[(4.85)^2 + 2(2.5)^3]^{1/5}$

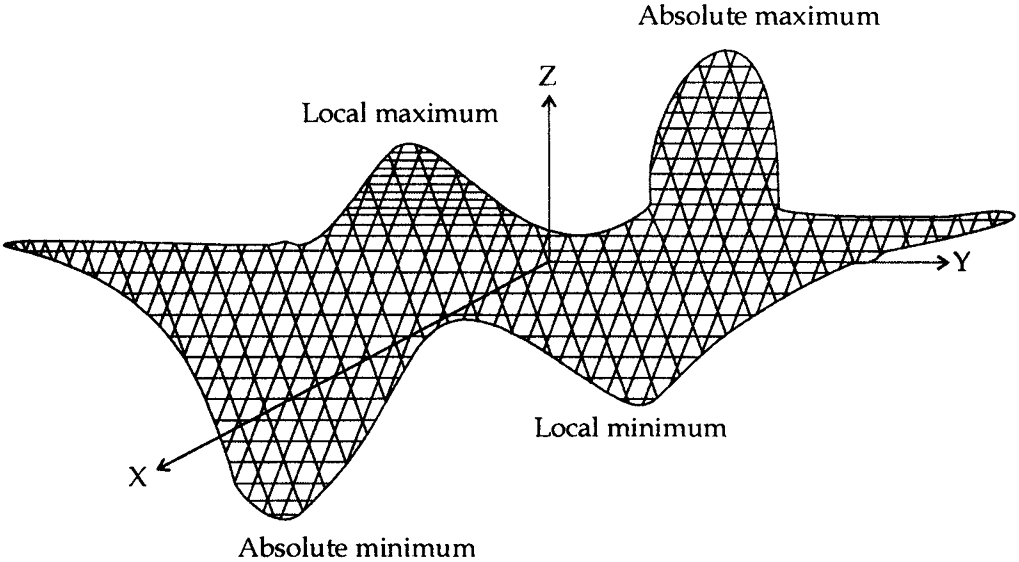
Ans. 2.15289

8. The focal length of a mirror is given by the formula $\frac{1}{v}-\frac{1}{u}=\frac{2}{f}$. If equal errors $\delta$ are made in the determination of u and v, show that the relative error $\frac{\delta f}{f}$ in the focal length is given by $\delta\left(\frac{1}{u}+\frac{1}{v}\right)$.

# Chapter 7

# Extrema of Functions of Several Variables

**Introduction :** There are many practical situations in which it is necessary or useful to know the largest and smallest values of a function of two variables. For example, if we consider the plot of a function f(x, y) of two variables to look like a mountain range, then the mountain tops, or the high points in their immediate vicinity, are called local maxima of f(x, y) and the valley bottom, or the low points in their immediate vicinity, are called local minima of f(x, y). The highest mountain and deepest valley in the entire mountain range are known as the absolute maxima and the absolute minimum respectively.



## Definitions

**Definition (i):** A function f of two variables has a local maximum at (a, b) if f(x, y) ≤ f(a, b) when (x, y) is in neighbourhood of (a, b). The number f(a, b) is called local maximum value of f.

If $f(x, y) \le f(a, b)$ for all points $(x, y)$ in the domain of the function f, then the function has its absolute maximum at $(a, b)$ and $f(a, b)$ is the absolute maximum value of f.

**Definition (ii)**

If $f(x, y) \ge f(a, b)$ when $(x, y)$ is near $(a, b)$ then $f(a, b)$ is the local minimum value of f. If $f(x, y) \ge f(a, b)$ for all points $(x, y)$ in the domain of f then f has its absolute minimum value at $(a, b)$.

**Theorem:** If f has a local maximum or minimum at (a, b) and the first order partial derivatives of f exist at a, b, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$

**Definition (iii):** A point (a, b) in the domain of a function f(x, y) is called a critical point (or stationary point) of f(x, y) if $f_x(a, b) = 0$ and $f_y(a, b) = 0$ or if one or both partial derivatives do not exist at (a, b).

**Definition (iv):** A point (a, b) where f(x, y) has neither a maximum nor a minimum is called a saddle point to f(x, y).

**Necessary Condition for Extremum Values of functions of two Variables:**

Let the sign of $f(a + h, b + k) - f(a, b)$ remain of the same for all values (positive or negative) of h, k. Then we have

(i) For $f(a + h, b + k) - f(a, b) < 0$, f (a, b) is maximum

(ii) For $f(a + h, b + k) - f(a, b) > 0$, f (a, b) is minimum

By Taylor's theorem for two variables, we have

$$f(a + h, b + k) = f(a, b) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{(a,b)} + \frac{1}{\underline{|2}}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial f}{\partial x\partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)} + \ldots\ldots.$$

or $f(a+h, b+k) - f(a, b) = \left(h\frac{\partial f}{\partial x}\right)_{(a,b)} + \left(k\frac{\partial f}{\partial y}\right)_{(a,b)} + \ldots\ldots\ldots\ldots\ldots$terms of second and higher orders in h and k (1)

By taking h and k to be sufficiently small, we can neglect the second and higher order terms. Thus, the first degree terms in h and k can be made to govern the sign of the left hand side of equation (1). Therefore the sign of [f (a+h, b+k) - f(a, b)] = the sign of $\left[h\left(\frac{\partial f}{\partial x}\right)_{(a,b)} + k\left(\frac{\partial f}{\partial y}\right)_{(a,b)}\right]$ (2)

Taking k =0, we find that if $\left(\frac{\partial f}{\partial x}\right)_{(a,b)} \ne 0$ the right hand side of equation (2) changes sign whenever h change sign. Therefore, f(x, y) cannot have a maximum or minimum at x =a, y = b if $\left(\frac{\partial f}{\partial x}\right)_{(a,b)} \ne 0$

similarly, taking h =0, we can find that f(x, y) cannot have a maximum or a minimum at x =a, y = b if $\left(\frac{\partial f}{\partial y}\right)_{(a,b)} \neq 0$

Thus, the necessary condition for f(x, y) to have a maximum or a minimum at x = a, y = b is

$$\left(\frac{\partial f}{\partial x}\right)_{(a,b)} = 0 \text{ and } \left(\frac{\partial f}{\partial y}\right)_{(a,b)} = 0$$

This condition is necessary but not sufficient for existence of maxima or minima.
Now, from equation (1)

$$f(a+h, b+k) - f(a, b) = \frac{1}{\underline{|2}}\left[h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right]$$

$$f(a+h, b+k) - f(a, b) = \frac{1}{\underline{|2}}[h^2 r + 2hks + k^2 t] \qquad (3)$$

where $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$ and $t = \frac{\partial^2 f}{\partial y^2}$ at (a, b)

since, we assumed that the sign of the LHS of equation (3) is the sign of the RHS of the equation (3) i.e.

sign of LHS of equation (3) = sign of $[rh^2 + 2hks + k^2 t]$

$= \text{sign of } \frac{1}{r}[r^2h^2 + 2hkrs + k^2rt]$

$= \text{sign of } \frac{1}{r}[(r^2h^2 + 2hkrs + k^2s^2) + (-k^2s^2 + k^2rt)]$

$= \text{sign of } \frac{1}{r}\left[(hr + ks)^2 + k^2(rt - s^2)\right]$

$= \text{sign of } \frac{1}{r}[k^2(rt - s^2)]$

$\because$ $(hr + ks)^2$ is always positive

= sing of r if $rt - s^2 > 0$

Hence, if $rt - s^2 > 0$, then f(x, y) has a maximum or a minimum at (a, b) according to $r < 0$ or $r > 0$ respectively.

**Working rule to find Extremum Values:**

The above proposition gives us the following rule for determining the maxima and minima of functions of two variables.

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ i.e. p and q equate then to zero. Solve these simultaneous equations for x and y Let $a_1, b_1; a_2, b_2; \ldots\ldots\ldots$ be the pairs of roots.

Find $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}$, (i.e. r, t and s respectively) and substitute in them by turns $a_1, b_1; a_2, b_2; \ldots\ldots\ldots$ for x, y. Calculate the value of $rt - s^2$ for each pair of roots.

If $rt - r^2 > 0$ and r is negative for a pair of roots, f(x,y) is a maximum for this pair. If $rt - r^2 > 0$ and r is positive, f(x, y) is a minimum. If $rt - s^2 < 0$, the function has a saddle point there.

If $rt - s^2 = 0$, the case is undecided, and further investigation is necessary to decide it.

**Example 1:** Find the local maxima or local minima of the function $f(x, y) = x^2 + y^2 - 2x - 6y + 14$

**Solution:** We have

$f(x,y) = x^2 + y^2 - 2x - 6y + 14$

Therefore

$f_x(x, y) = 2x - 2$ and $f_y(x, y) = 2y - 6$

If we put $f_x(x, y) = 0$ we have

$2x - 2 = 0$

$\Rightarrow x = 1$

If we put $f_y(x,y) = 0$, we have

$2y - 6 = 0$

$\Rightarrow y = 3$

we get x =1 and y = 3 as critical points of function f(x, y). Now, we can write f(x, y) as

$f(x, y) = 4 + (x - 1)^2 + (y - 3)^2$

since $(x - 1)^2 \geq 0$ and $(y - 3)^2 \geq 0$, we have $f(x, y) \geq 4$ for all values of x and, therefore, f(1, 3) =4 is a local minimum. It is also the absolute minimum of f.

**Example 2:** Find the extreme value of $f(x,y) = y^2 - x^2$

**Solution:** we have

$f(x, y) = y^2 - x^2$

Differentiating partially with respect to x and y, we get

$f_x = -2x, f_y = 2y$

which follows fx(0,0) =0 and fy (0,0)=0

Therefore, (0,0) is the only critical point.

However the function f has neither local maximum nor a local minima at (0,0). Thus (0,0) is called saddle point of f.

**Example 3:** Find the maximum and minimum of the function $f(x,y) = x^3 + y^3 - 3axy$

(U.P.T.U. 2004, 2006; U.P.P.C.S. 1992; B.P.S.C. 1997)

**Solution:** Given that

$f(x,y) = x^3 + y^3 - 3\,axy$ (i)

Therefore

$$p = \frac{\partial f}{\partial x} = 3x^2 - 3ay,\ q = \frac{\partial f}{\partial y} = 3y^2 - 3ax$$

$$r = \frac{\partial^2 f}{\partial x^2} = 6x,\ s = \frac{\partial^2 f}{\partial x \partial y} = -3a,\ t = \frac{\partial^2 f}{\partial y^2} = 6y$$

for maxima and minima, we have

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 - 3ay = 0 \Rightarrow x^2 = ay \qquad \text{(ii)}$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow 3y^2 - 3ax = 0 \Rightarrow y^2 = ax \qquad \text{(iii)}$$

putting the value of y in equation (iii), we get

$x^4 = a^3 x$

$\Rightarrow x\,(x^3 - a^3) = 0$

$\Rightarrow x\,(x - a)\,(x^2 + ax + a^2) = 0$

$\Rightarrow x = 0,\ x = a$

Putting $x = 0$ in (ii), we get $y = 0$, and putting $x = a$ in (ii) we get $y = a$. Therefore (0,0) and (a, a) are the stationary points (i.e. critical points) testing at (0, 0).

$r = 0,\ t = 0,\ s = -3a \Rightarrow rt - s^2 =$ Negative

Hence there is no extemum value at (0,0).

Testing at (a, a)

$r = 6a,\ t = 6a,\ s = -3a$

$\Rightarrow rt - s^2 = 6a \times 6a - (-3a)^2$

$= 36a^2 - 9a^2$

$= 27a^2 > 0$ and also $r = 6a > 0$

Therefore (a, a) is a minimum point.

The minimum value of $f(a, a) = a^3 + a^3 - 3a^3$

$= -a^3$ Answer.

**Example 4:** Test the function $f(x, y) = x^3 y^2 (6 - x - y)$ for maxima and minima for points not at the origin.

**Solution:**

Here $f(x,y) = x^3 y^2 (6 - x - y)$

$= 6x^3y^2 - x^4y^2 - x^3y^3$

$$\therefore p = \frac{\partial f}{\partial x} = 18x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$q = \dfrac{\partial f}{\partial y} = 12x^3y - 2x^4y - 3x^3y^2$

$r = \dfrac{\partial^2 f}{\partial x^2} = 36xy^2 - 12x^2y^2 - 6xy^3$

$= 6xy^2(6 - 2x - y)$

$s = \dfrac{\partial^2 f}{\partial x \partial y} = 36x^2y - 8x^3y - 9x^2y^2$

$= x^2y(36 - 8x - 9y)$

$t = \dfrac{\partial^2 f}{\partial y^2} = 12x^3 - 2x^4 - 6x^3y$

$= x^3(12 - 2x - 6y)$

Now, for maxima or minima, we have

$\dfrac{\partial f}{\partial x} = 0 \Rightarrow x^2y^2(18 - 4x - 3y) = 0$ (i)

and $\dfrac{\partial f}{\partial y} = 0$

$\Rightarrow x^3y(12 - 2x - 3y) = 0$ (ii)

From (i) & (ii)

$4x + 3y = 18$

$2x + 3y = 12$ and $x = 0 = y$

solving, we get $x = 3$, $y = 2$ and $x = 0 = y$. Leaving $x = 0 = y$, we get $x = 3$, $y = 2$. Hence (3, 2) is the only stationary point under consideration.

Now,

$rt - s^2 = 6x^4y^4(6 - 2x - y)(12 - 2x - 6y) - x^4y^2(36 - 8x - 9y)^2$

At (3, 2)

$rt - s^2 = +$ ive ($> 0$)

Also, $r = 6(3)(4)(6 - 6 - 4) = -$ ive ($< 0$)

$\therefore f(x,y)$ has a maximum value at (3, 2).

**Example 5:** Examine for minimum and maximum values. $\sin x + \sin y + \sin(x + y)$

(U.P.P.C.S. 1991)

**Solution:** Here, $f(x,y) = \sin x + \sin y + \sin(x + y)$

$\therefore p = \dfrac{\partial f}{\partial x} = \cos x + \cos(x + y)$

$q = \dfrac{\partial f}{\partial y} = \cos y + \cos(x + y)$

$r = \dfrac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x + y)$

$s = \frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y)$

$t = \frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y)$

Now $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$\Rightarrow \cos x + \cos(x+y) = 0$...........(i) and $\cos y + \cos(x+y) = 0$ ........(ii)
subtracting (ii) from (i) we have
$\cos x - \cos y = 0$ or $\cos x = \cos y$
$\Rightarrow x = y$
From (i), $\cos x + \cos 2x = 0$
or $\cos 2x = -\cos x = \cos(\pi - x)$
or $2x = \pi - x \therefore x = \frac{\pi}{3}$

$\therefore x = y = \frac{\pi}{3}$ is a stationary point

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, $r = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}$, $s = \frac{\sqrt{3}}{2}$

$t = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}$

$\therefore rt - s^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$

Aslo $r < 0$

$\therefore f(x, y)$ has a maximum value at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

Maximum value $= f\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$= \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\frac{2\pi}{3}$

$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$ Answer.

**Example 6:** Test the function $f(x,y) = (x^2 + y^2)e^{-(x^2+y^2)}$ for maxima or minima for point not on the circle $x^2 + y^2 = 1$

**Solution:**

Here $f(x, y) = (x^2 + y^2)\, e^{-(x^2+y^2)}$

$\therefore \dfrac{\partial f}{\partial x} = (x^2 + y^2)\ e^{-(x^2+y^2)}\ (-2x) + 2x\ e^{-(x^2+y^2)}$

$= 2x\ (1 - x^2 - y^2)\ e^{-(x^2+v^2)}$

$\dfrac{\partial f}{\partial y} = (x^2 + y^2)\ e^{-(x^2+y^2)}\ (-2y) + 2y\ e^{-(x^2+y^2)}$

$= 2y\ (1 - x^2 - y^2)\ e^{-(x^2+y^2)}$

Now for maxima and minima, we have

$\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$

$\Rightarrow\ 2x\ (1 - x^2 - y^2)\ e^{-\left(x^2 + y^2\right)} = 0$ and $2y\ (1 - x^2 - y^2)\ e^{-\left(x^2 + v^2\right)} = 0$

$\Rightarrow\ x = 0,\ y = 0$ and $x^2 + y^2 = 1$

Leaving $x^2 + y^2 = 1$ as given, take $x = 0,\ y = 0$

Hence (0,0) is the only stationary point

$f_x = (2x - 2x^3 - 2xy^2)\ e^{-(x^2+y^2)}$

$r = \dfrac{\partial^2 f}{\partial x^2} = (2 - 6x^2 - 2y^2)\ e^{-(x^2+y^2)} + (2x - 2x^3 - 2xy^2)\ e^{-(x^2+y^2)}\ (-2x)$

$= e^{-(x^2+y^2)}\ (4x^4 - 10\ x^2 + 4x^2y^2 - 2y^2 + 2)$

$s = f_{xy} = (-\ 4xy)\ e^{-(x^2+y^2)} + (2x - 2x^3 - 2xy^2)\ e^{-(x^2+y^2)}\ (-2y)$

$= (-\ 8xy + 4\ x^3y + 4xy^3)\ e^{-(x^2+y^2)}$

$t = \dfrac{\partial^2 f}{\partial y^2} = e^{-(x^2+y^2)}\ (2 - 2x^2 - 6y^2) + e^{-(x^2+y^2)}\ .\ (2y - 2yx^2 - 2y^3)\ (-2y)$

$= e^{-(x^2+y^2)}\ (2 - 2x^2 - 10y^2 - 4x^2y^2 + 4y^4)$

At (0,0), $r = 0$, $s = 0$, $t = 2$

$rt - s^2 = 4 > 0$

Also $r = 2\ (> 0)$

$\therefore$ f(x, y) has a minimum value at (0,0)

$\therefore$ Minimum value $= (0+0)\ e^{-0} = 0$ Answer.

**Example 7:** A rectangular box, open at the top, is to have a given capacity. Find the dimensions of the box requiring least material for its construction.

**Solution:** Let x, y and z be the length, breadth and height respectively, let V be the given capacity and S, the surface

V is given ⇒ V is constant

$V = xyz$

or $Z = \dfrac{V}{xy}$ (i)

$S = xy + 2xz + 2yz$

$$= xy + \frac{2V}{y} + \frac{2V}{x} = f(x,y) \text{ using (i)}$$

$$\therefore p = \frac{\partial f}{\partial x} = y - \frac{2V}{x^2},\ q = \frac{\partial f}{\partial y} = x - \frac{2V}{y^2}$$

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{4V}{x^3},\ s = \frac{\partial^2 f}{\partial x \partial y} = 1$$

$$t = \frac{\partial^2 f}{\partial y^2} = \frac{4V}{y^3}$$

Now $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$$\Rightarrow y - \frac{2V}{x^2} = 0 \text{ ..........(ii) and } x - \frac{2V}{y^2} = 0 \text{ ..........(iii)}$$

From (i) $y = \frac{2V}{x^2}$

$\therefore$ From (ii), $x - 2V\ \frac{x^4}{4V^2} = 0$

or $x\left(1 - \frac{x^3}{2V}\right) = 0$

or $x = (2V)^{1/3}$ (As $x \neq 0$)

and $y = \frac{2V}{x^2} = \frac{2V}{(2V)^{3/2}} = (2V)^{1/3}$

$\therefore$ $x = y = (2V)^{1/3}$ is a stationary point. At this point, $r = \frac{4V}{2V} = 2 > 0$, $s = 1$,

$t = \frac{4V}{2V} = 2$

So that $rt - s^2 = 4 - 1 = 3 > 0$ and $r > 0$

$\Rightarrow$S is minimum when $x = y = (2V)^{1/3}$

Also $z = \frac{V}{xy} = \frac{V}{(2V)^{2/3}}$

$$= \frac{V^{1/3}}{2^{2/3}} = \frac{(2V)^{1/3}}{2}$$

$$= \frac{y}{2}$$

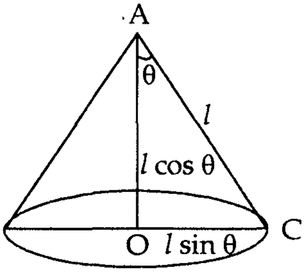
Hence S is minimum when $x = y = 2z = (2V)^{1/3}$ Answer.

**Example 8:** Find the semi-vertical angle of the cone of maximum volume and of a given slant height.

(U.P.T.U. 2005)

**Solution:** Let $l$ be the slant height and $\theta$ be the semi - vertical angle of the cone. Then, radius of the base, $r = OC = l\sin\theta$ and height of the cone, $h = OA = l\cos\theta$.

Let V be the volume of the cone then $V = \frac{1}{3}\pi r^2 h$



$$= \frac{1}{3}\pi(l\sin\theta)^2(l\cos\theta)$$

$$= \frac{1}{3}\pi\, l^3\sin^2\theta\,\cos\theta$$

$$\therefore \frac{dV}{d\theta} = \frac{1}{3}\pi\, l^2\{\sin^2\theta\,(-\sin\theta) + \cos\theta.\,2\sin\theta\cos\theta\}$$

$$= \frac{1}{3}\pi\, l^3\sin\theta\,(2\cos^2\theta - \sin^2\theta)$$

$$\frac{d^2V}{d\theta^2} = \frac{1}{3}\pi\, l^3\,[\sin\theta\,(-4\cos\theta\sin\theta - 2\sin\theta\cos\theta) + (2\cos^2\theta - \sin^2\theta)\cos\theta]$$

$$= \frac{1}{3}\pi\, l^3\,[-6\sin^2\theta\cos\theta + \cos\theta\,(2\cos^2\theta - \sin^2\theta)]$$

For max and min. $\frac{dV}{d\theta} = 0$

Which gives either $\sin\theta = 0$ or $\tan^2\theta = 2$

so that $\theta = 0$ or $\theta = \pm\tan^{-1}\sqrt{2}$

when $\theta = 0$, volume of cone becomes zero and the cone becomes a straight line which is not the case, when $\theta = \tan^{-1}\sqrt{2}$, we have

$$\frac{d^2V}{d\theta^2} = \frac{1}{3}\pi l^3\left[-6.\frac{2}{3}.\frac{1}{\sqrt{3}} + 0\right]$$

= - ive

Hence the volume of cone is maximum when $\theta = \tan^{-1}\sqrt{2}$.

When $\theta = -\tan^{-1}\sqrt{2}$, volume of cone becomes negative which is meaningless, hence is not the case. Answer.

**Example 9:** Find the volume of largest parallelopiped that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

(U.P.T.U. 2001)

**Solution:** Let (x,y,z) denote the co-ordinates of one vertices of the parallelopied which lies in the positive octant and V denote its volume so that

$V = 8xyz$ As 2x, 2y and 2z be the length, breadth and height respectively

$$\therefore V^2 = 64x^2y^2z^2$$

$$= 64\,x^2y^2c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$$

$$= 64c^2\left(x^2y^2-\frac{x^4y^2}{a^2}-\frac{x^2y^4}{b^2}\right) = f(x,y) \text{ say}$$

$$\Rightarrow \frac{\partial f}{\partial x} = 64c^2\left(2xy^2-\frac{4x^3y^2}{a^2}-\frac{2xy^4}{b^2}\right)$$

and $\frac{\partial f}{\partial y} = 64c^2\left(2x^2y-\frac{2x^4y}{a^2}-\frac{4x^2y^3}{b^2}\right)$

Now putting $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$ we get

$$\Rightarrow 2xy^2-\frac{4x^3y^2}{a^2}-\frac{2xy^4}{b^2}=0$$

$$\Rightarrow 1-\frac{2x^2}{a^2}-\frac{y^2}{b^2}=0 \qquad \text{(i)}$$

and $2x^2y-\frac{2x^4y}{a^2}-\frac{4x^2y^3}{b^2}=0$

$$\Rightarrow 1-\frac{x^2}{a^2}-\frac{2y^2}{b^2}=0 \qquad \text{(ii)}$$

Now multiply (i) by 2, we have

$$2-\frac{4x^2}{a^2}-\frac{2y^2}{b^2}=0 \qquad \text{(iii)}$$

subtracting (iii) from (ii) we have

$$-1+\frac{3x^2}{a^2}=0 \Rightarrow 3x^2=a^2$$

$$\therefore x=\frac{a}{\sqrt{3}} \text{ and } y=\frac{b}{\sqrt{3}}$$

Now $r = \dfrac{\partial^2 f}{\partial x^2} = 64c^2\left[2y^2 - \dfrac{12x^2y^2}{a^2} - \dfrac{2y^4}{b^2}\right]$

$s = \dfrac{\partial^2 f}{\partial x \partial y} = 64c^2\left[4xy - \dfrac{8x^3y}{a^2} - \dfrac{8xy^3}{b^2}\right]$

$t = \dfrac{\partial^2 f}{\partial y^2} = 64c^2\left[2x^2 - \dfrac{2x^4}{a^2} - \dfrac{12x^2y^2}{b^2}\right]$

and $rt - s^2 = (64\,c^2)\left(2y^2 - \dfrac{2y^4}{b^2} - \dfrac{12x^2y^2}{a^2}\right)(64c^2)\left(2x^2 - \dfrac{2x^4}{a^2} - \dfrac{12x^2y^2}{b^2}\right)$

$-\left[(64c^2)\left(4xy - \dfrac{8x^3y}{a^2} - \dfrac{8xy^3}{b^2}\right)\right]^2$

At $\left(\dfrac{a}{\sqrt{3}}, \dfrac{b}{\sqrt{3}}\right)$, $rt - s^2 > 0$ and $r < 0$

Hence f(x,y) is max at $\left(\dfrac{a}{\sqrt{3}}, \dfrac{b}{\sqrt{3}}\right)$

$$\Rightarrow V_{max}^2 = 64c^2\left[\frac{a^2}{3}.\frac{b^2}{3} - \frac{1}{a^2}.\frac{a^4}{9}.\frac{b^2}{3} - \frac{1}{b^2} \times \frac{a^2}{3} \times \frac{b^4}{9}\right]$$

$$= \frac{64a^2b^2c^2}{27}$$

$$\Rightarrow V_{max} = \frac{8abc}{3\sqrt{3}}$$ Answer.

# Chapter 8

# Lagrange's Method of Undetermined Multipliers

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In many problems, a function of two or more variables is to be optimized, subjected to a restriction or constraint on the variables, here we will consider a function of three variables to study Lagrange's method of undetermined multipliers.

Let

$u = f(x, y, z)$ (i)

be a function of three variables connected by the relation

$\phi(x, y, z) = 0$ (ii)

The necessary conditions for u to have stationary values are

$$\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0, \frac{\partial u}{\partial z} = 0$$

Differentiating equation (i), we get du =0 i.e.

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0 \qquad \text{(iii)}$$

Differentiating equation (ii) we get dϕ =0 i.e.

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0 \qquad \text{(iv)}$$

Multiplying equation (iv) by λ and adding to equation (iii) we get

$$\left(\frac{\partial u}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial u}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz = 0$$

This equation will be satisfied if

$$\frac{\partial u}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \qquad \text{(v)}$$

$$\frac{\partial u}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \qquad \text{(vi)}$$

$$\frac{\partial u}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \qquad \text{(vii)}$$

where $\lambda$ is the Lagrange multiplier. On solving equations (ii), (v) (vi) and (vii), we get the values of x, y, z and $\lambda$ which determine the stationary points and hence the stationary values of f(x, y, z).

**Note:** (i) Lagrange's method gives only the stationary values of f(x, y, z). The nature of stationary points cannot be determined by this method.

(ii) If there are two constraints $\phi_1(x, y, z) = 0$ & $\phi_2(x, y, z) = 0$, then the auxiliary function is $F(x, y, z) = f(x, y, z) + \lambda_1, \phi, (x, y, z) + \lambda_2 \phi_2 (x, y, z)$ here $\lambda_1$ and $\lambda_2$ are the two Lagrange multipliers. The stationary values are obtained by solving the five equations $F_x = 0$, $F_y = 0$, $F_z = 0$, $F_{\lambda_1} = 0$ and $F_{\lambda_2} = 0$

**Example 10:** Find the volume of largest parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ using Lagrange's method of Multipliers.

(I.A.S. 2007; U.P.T.U. 2001, 2003)

**Solution:** Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$\phi(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \qquad \text{(i)}$$

Let 2x, 2y, and 2z be the length, breadth and height, respectively of the rectangular parallelopiped inscribed in the ellipsoid. Then

$V = (2x)(2y)(2z) = 8xyz$

Therefore, we have

$$\frac{\partial V}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \Rightarrow 8yz + \lambda \frac{2x}{a^2} = 0 \qquad \text{(ii)}$$

$$\frac{\partial V}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \Rightarrow 8xz + \lambda \frac{2x}{b^2} = 0 \qquad \text{(iii)}$$

$$\frac{\partial V}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \Rightarrow 8xy + \lambda \frac{2z}{c^2} = 0 \qquad \text{(iv)}$$

Multiplying (ii), (iii) and (iv) be x, y and z respectively, and adding, we get

$$24\,xyz + 2\lambda \left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right] = 0$$

$24\,xyz + 2\lambda\,(1) = 0$

$\Rightarrow \lambda = -12\,xyz$

putting the value of $\lambda$ in (ii) we have

$$8yz + (-12\,xyz)\,\frac{2x}{a^2} = 0$$

$$\Rightarrow 1-\frac{3x^2}{a}=0$$

$$\Rightarrow x=\frac{a}{\sqrt{3}}$$

Similarly, on putting λ = - 12 xyz in equation (iii) and (iv) we get

$$y=\frac{b}{\sqrt{3}}, z=\frac{c}{\sqrt{3}}$$

Hence, the volume of the largest parallelopiped = 8xyz

$$=8\left(\frac{a}{\sqrt{3}}\right)\left(\frac{b}{\sqrt{3}}\right)\left(\frac{c}{\sqrt{3}}\right)$$

$$=\frac{8abc}{3\sqrt{3}}$$ Answer.

**Example 11:** Find the extreme value of $x^2 + y^2 + z^2$, subjected to the condition xy + yz + zx = p.

(U.P.T.U. 2008)

**Solution:** Let $f = x^2 + y^2 + z^2$ and $\phi = xy + yx + zx - p$.

Then for maximum or minimum, we have

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0 \Rightarrow 2x+\lambda(y+z)=0 \qquad \text{(i)}$$

$$\frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0 \Rightarrow 2y+\lambda(x+z)=0 \qquad \text{(ii)}$$

$$\frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0 \Rightarrow 2z+\lambda(x+y)=0 \qquad \text{(iii)}$$

Multiplying equation (i) by x, equation (ii) by y and equation (iii) by z and adding we get.

$$2(x^2 + y^2 + z^2) + 2\lambda (xy + yz + zx) =0$$

$$\Rightarrow 2f + 2\lambda p =0$$

$$\text{or } \lambda=-\frac{f}{p}$$

on putting this value of λ in (i) (ii) and (iii), we get

$$2x - \frac{f}{p}(y + z) =0$$

$$2y - \frac{f}{p}(x + z) =0$$

$$2z - \frac{f}{p}(x + y) =0$$

or $2x - 2y + \frac{f}{p}(x - y) = 0$

$\left(\frac{f}{p} + 2\right)(x - y) = 0$

$\frac{f}{p} = -2$ and $x = y$

Similarly, we get $y = z$, therefore, $f = 3x^2$

But

$xy + yz + zx = p$

$\Rightarrow 3x^2 = p \Rightarrow x^2 = p/3$

Therefore, extrema occur if

$x^2 + y^2 + z^2 = p$ Answer.

**Example 12:** Show that the rectangular solid of maximum volume that can be inscribed in a given sphere is a cube.

(U.P.T.U. 2004)

**Solution:** Let 2x, 2y, 2z be the length, breadth and height of the rectangular solid and r be the radius of the sphere

Then $x^2 + y^2 + z^2 = R^2$ (i)

Volume $V = 8xyz$ (ii)

Consider Lagrange's function

$F(x,y,z) = 8xyz + \lambda(x^2 + y^2 + z^2 - R^2)$

For stationary values,

$dF = 0$

$\Rightarrow \{8yz + \lambda(2x)\}dx + \{8xz + \lambda(2y)\}dy + \{8xy + \lambda(2z)\}dz = 0$

$\Rightarrow 8yz + 2\lambda x = 0$ (iii)

$8zx + 2\lambda y = 0$ (iv)

$8xy + 2\lambda z = 0$ (v)

From (iii) $2\lambda x^2 = -8xyz$

From (iv) $2\lambda y^2 = -8xyz$

From (v) $2\lambda z^2 = -8xyz$

$\therefore 2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2$

or $x^2 = y^2 = z^2$ or $x = y = z$

Hence rectangular solid is a cube. Hence Proved.

**EXERCISE**

1. Find all relative extrema and saddle points of the function.

$f(x,y) = 2x^2 + 2xy + y^2 - 2x - 2y + 5$

2. A rectangular box, open at the top, is to have a volume of 32 cc. Find the dimensions of the box requiring the least material for its construction.

(U.P.T.U. 2005)

Ans. $l = 4$, $b = 4$ and $h = 2$ units

3. Divide 24 into three parts such that the continued product of the first part, the square of the second part, and the cube of the third part is maximum.

Ans. $x = 12, y = 8, z = 4$

4. Show that the rectangular solid of maximum volume that can be inscribed in a given sphere is a cube.

(U.P.T.U. 2003)

5. The sum of three positive number is constant. Prove that their product is maximum when they are equal.

6. Find the maximum and minimum distances of the point (3, 4, 12) from the sphere.

Ans. Minimum distance = 12
Maximum distance = 14

7. The temperature T at any point (x, y, z) in space is $T = 400\,xyz^2$. Find the highest temperature at the surface of a unit sphere $x^2 + y^2 + z^2 = 1$

Ans. 50

8. A tent of given volume has a square base of side 2a and has its four sides of height b vertical and is surmounted by a pyramid of height h. Find the values of a and b in terms of h so that the canvas required for its construction be minimum.

Ans. $a = \frac{\sqrt{5}}{2}h$

and $b = \frac{h}{2}$

9. Prove that the stationary values of $u = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$

where $lx + my + nz = 0$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ are the roots of the equation.

$$\frac{l^2a^4}{1-a^2u} + \frac{m^2b^4}{1-b^2u} + \frac{n^2c^4}{1-c^2u} = 0$$

(B.P.S.C. 2007)

10. Use Lagrange's method of undetermined multipliers to find the minimum value of $x^2 + y^2 + z^2$ subjected to the conditions.

$x + y + z = 1, \quad xyz + 1 = 0$

Ans. 3

11. Find the minimum value of $x^2 + y^2 + z^2$, given that $ax + by + cz = p$

Ans. $\frac{p^2}{a^2+b^2+c^2}$

12. Find the maximum and minimum distances from the origin to the curve.
$x^2 + 4xy + 6y^2 = 140$

[U.P.T.U. (CO) 2003]

Ans. Maximum distance = 21.6589
Minimum distance = 4.5706

13. If $u = ax^2 + by^2 + cz^2$ and $x^2 + y^2 + z^2 = 1$ and $lx + my + nz = 0$ are the constraints, Prove

$$\frac{l^2}{a-u}+\frac{m^2}{b-u}+\frac{n^2}{c-u}=0$$

**OBJECTIVE PROBLEMS**

Four alternative answers are given for each question, only one of them is correct, tick mark the correct.

1. Maxima and minima occur -
(i) Simultaneously (ii) Once
(iii) Alternately (iv) rarely

Ans. (iii)

2. The minimum value of $|x^2 - 5x + 2|$ is

(M.P.P.C.S. 1991)

(i) -5 (ii) 0
(iii) -1 (iv) -2

Ans. (ii)

3. The maximum value of $\frac{1}{\sqrt{2}}(\sin x - \cos x)$ is

(M.P.P.C.S. 1991)

(i) 1 (ii) $\sqrt{2}$
(iii) $\frac{1}{\sqrt{2}}$ (iv) 3

Ans. (i)

4. The triangle of maximum area inscribed in a circle of radius r is

(I.A.S. 1993)

(i) A right angled triangle with hypotenuse measuring 2r
(ii) An equilarteral triangle
(iii) An isosceles triangle of height r
(iv) Does not exist

Ans. (ii)

5. Let $f(x) = |x|$, then

(I.A.S. 1993)

(i) $f'(0) = 0$
(ii) $f(x)$ is maximum at $x = 0$
(iii) $f(x)$ is minimum at $x = 0$
(iv) None of the above.

Ans. (iv)

(I.A.S. 1990)

6. If $y = a \log x + bx^2 + x$ has its extremum value at $x = -1$ and $x = 2$ then

(i) $a = 2, b = -1$ (ii) $a = 2, b = -\frac{1}{2}$

(iii) $a = -2, b = \frac{1}{2}$ (iv) None of these

Ans. (iii)

7. The function $f(x) = x^3 - 6x^2 + 24x + 4$ has

(I.A.S. 1990)

(i) A maximum value of $x = 2$
(ii) A minimum value of $x = 2$
(iii) A maximum value at $x = 4$ and minimum value at $x = 6$
(iv) Neither maximum nor minimum at any point

Ans. (iv)

8. The function $\sin x (1 + \cos x)$ is maximum in the interval $(0, \pi)$, when -

(U.P.P.C.S. 1994, M.P.P.C.S. 1995)

(i) $x = \pi/4$ (ii) $x = \pi/2$
(iii) $x = \frac{\pi}{3}$ (iv) $x = \frac{2\pi}{3}$

Ans. (iii)

9. A necessary condition for $f(a)$ to be an extreme value of $f(x)$ is that

(R.A.S. 1995)

(i) $f(a) = 0$ (ii) $f'(0) = 0$
(iii) $f''(a) = 0$ (iv) $f'''(a) = 0$

Ans. (ii)

10. The value of function

$$f(x) = x + \frac{1}{x}$$

at the points of minimum and maximum one respectively

(M.P.P.C.S. 1995)

(i) -2 and 2 (ii) 2 and -2
(iii) -1 and 1 (iv) 1 and -1

Ans. (iv)

11. The profit function is

$P(x) = -\frac{1}{2}x^2 + 32x - 480$

then the profit is maximum if the number of item (x) produced and sold is

(U.P.P.C.S. 1991)

(i) 18 (ii) 17
(iii) 24 (iv) 32

Ans. (iv)

12. The value of x for which the function $f(x) = x^{1/x}$ has a maximum is given by

(R.A.S. 1993)

(i) $x = e$ (ii) $x = \frac{1}{e}$
(iii) $x = -e$ (iv) $x = -\frac{1}{e}$

Ans. (i)

13. The maximum value of $\frac{1}{x^{\lambda}}$ is

(M.P.P.C.S. 1994)

(i) e (ii) $e^{-e}$
(iii) $e^{-1/e}$ (iv) $e^{1/e}$

Ans. (iv)

14. If $x + y = k$, $x > 0$, $y > 0$, then xy is the maximum when

(M.P.P.C.S. 1991)

(i) $x = ky$ (ii) $kx = y$
(iii) $x = y$ (iv) None of these

Ans. (iv)

15. Maximum value of $a \sin\theta + b \cos\theta$ is

(M.P.P.C.S. 1992)

(i) $\sqrt{a^2 + b^2}$ (ii) $\sqrt{a + b}$
(iii) $\sqrt{a - b}$ (iv) $2\sqrt{a^2 + b^2 + a + b}$

Ans. (i)

16. Maximum value of sin x + cos x is

(M.P.P.C.S. 1992)

(i) 2 (ii) $\sqrt{2}$
(iii) 1 (iv) $1 + \sqrt{2}$

Ans. (ii)

17. If the functions u, v, w of three independent variables x, y, z are not independent, then the Jacobian of u, v, w with respect to x, y, z is always equal to

(I.A.S. 1995)

(i) 1 (ii) 0
(iii) The Jalonbian of x, y, z w.r.t u, v, w
(iv) infinity

Ans. (ii)

18. The maximum rectangle, inscribed in a circle of radius 1, is of area

(U.P.P.C.S. 1995)

(i) 1 (ii) 2
(iii) 4 (iv) 8

Ans. (ii)

19. The derivative $f'(x)$ of a function $f(x)$ is positive or zero in (a, b) without always being zero. Then, which of the following is true in (a, b)

(i) $f(b) < f(a)$ (ii) $f(a) < f(b)$
(iii) $f(b) - f(a) = f'(c)$ (iv) $f(b) = f(a)$

Ans. (ii)

20. The conditions for f(x, y) to be maximum are

(i) $rt - s^2 > 0, r > 0$ (ii) $rt - s^2 > 0, r < 0$
(iii) $rt - s^2 < 0, r > 0$ (iv) $rt - s^2 < 0, r < 0$

Ans. (ii)

21. The conditions for f(x, y) to be minimum are

(i) $rt - s^2 > 0, r < 0$ (ii) $rt - s^2 > 0, r > 0$
(iii) $rt - s^2 < 0, r > 0$ (iv) $rt - s^2 < 0, r < 0$

Ans. (ii)

22. The stationary points of f(x, y) given by -

(i) $f_x = 0, f_y = 0$ (ii) $f_x \neq 0, f_y = 0$
(iii) $f_y \neq 0, f_{(x)} = 0$ (iv) None of these

Ans. (i)

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# UNIT - 3

# Matrices

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# Chapter 9
# Matrices

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**INTRODUCTION :** Today, the subject of matrices is one of the most important and powerful tool in mathematics which has found application to a large number of disciplines such as Engineering, Economics, Statistics, Atomic Physics, Chemistry, Biology, Sociology etc. Matrices are a powerful tool in modern mathematics. Matrices also play an important role in computer storage devices. The algebra and caluclus of matrices forms the basis for methods of solving systems of linear algebraic equations, for solving systems of linear differential equations and for analysis solutions of systems of nonlinear differential equations.

**Determinant of a matrix : -** Every square matrix A with numbers as elements has associated with it a single unique number called the determinant of A, which is written det A. if A is n × n, the determinant of A is indicated by displaying the elements $a_{ij}$ of A between vertical bars

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \dots\dots\dots a_{1n} \\ a_{21} & a_{22} & \dots\dots\dots a_{2n} \\ \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots\dots\dots a_{nn} \end{vmatrix}$$

The number n is called the order of determinant A.

**Non Singular and singular Matrices :**

A square matrix A = $[a_{ij}]$ is said to be non singular according as $|A| \neq 0$ or $|A| = 0$

**Adjoint of a matrix :** Let A =[aij] be a square matrix. Then the adjoint of A, denoted by adj (A), is the matrix given by adj (A) = $\left[A_{ij}\right]^{T}_{n\times n}$, where $A_{ij}$ is the cofactor of $a_{ij}$ in A, i.e. if

$$A = \begin{vmatrix} a_{11} & a_{12} & \dots\dots\dots a_{1n} \\ a_{21} & a_{22} & \dots\dots\dots a_{2n} \\ \dots & \dots & \dots\dots\dots\dots \\ a_{n1} & a_{n2} & \dots\dots\dots a_{nn} \end{vmatrix}$$

$$\text{Then adj}(A) = \begin{bmatrix} A_{11} & A_{12} & \ldots\ldots\ldots\ldots A_{n1} \\ A_{21} & A_{22} & \ldots\ldots\ldots\ldots A_{n2} \\ \cdots & \cdots & \ldots\ldots\ldots\ldots\ldots \\ A_{1n} & A_{2n} & \ldots\ldots\ldots\ldots A_{nn} \end{bmatrix}$$

**Inverse of a matrix :** Let A be square matrix of order n. Then the matrix B of order n if it exists, such that

$AB = BA = I_n$

is called the inverse of A and denoted by $A^{-1}$ we have

$A(\text{adj}A) = |A|I$

or $A\frac{(\text{adj}A)}{|A|} = I$, Provided $|A| \neq 0$

or $A^{-1} = \frac{\text{adj}A}{|A|}$, if $|A| \neq 0$

**Theorem :** The inverse of a matrix is unique.

**Proof :** Let us consider that B and C are two inverse matrices of a given matrix, say A

Then $AB = BA = I$ $\quad\therefore$ B is inverse of A

and $AC = CA = I$ $\quad\therefore$ C is inverse of A

$C(AB) = (CA)B$ $\quad$ by associative law

or $CI = IB$

or $C = B$

Thus, the inverse of matrix is unique.

**Existence of the Inverse : Theorem:** A necessary and sufficient condition for a square matrix A to possess the inverse is that $|A| \neq 0$.

(I.A.S. 1973)

**Proof :** The condition is necessary

Let A be an n × n matrix and let B be the inverse of A.

Then $AB = I_n$

$\therefore |AB| = |I_n| = I$

$\therefore |A|\,|B| = 1 \qquad \therefore |AB| = |A|\,|B|$

$\therefore |A|$ must be different from 0.

Conversely, the condition is also sufficient.

If $|A| \neq 0$, then let us define a matrix B by the relation

$$B = \frac{1}{|A|}(\text{adj}.A)$$

$$\text{Then } AB = A\left(\frac{1}{|A|}\text{adj}.A\right)$$

$$= \frac{1}{|A|}(\text{adj.}A) = \frac{1}{|A|}|A|I_n = I_n$$

Similarly $BA = \left(\frac{1}{|A|}\text{adj.}A\right)A = \frac{1}{|A|}(\text{adj.}A)A$

$$= \frac{1}{|A|}.|A|I_n = I_n$$

Thus $AB = BA = I_n$

Hence, the matrix A is invertible and B is the inverse of A.

**Reversal law for the inverse of a Product, Theorem :** If A, B be two n rowed non-singular matrices, then AB is also non - singular and

$(AB)^{-1} = B^{-1} A^{-1}$

i.e. the inverse of a product is the product of the inverse taken in the reverse order.

(I.A.S. 1969, U.P.P.C.S. 1995)

**Proof :** Let A and B be two n rowed non singular matrices,

We have $|AB| = |A|\,|B|$

Since $|A| \neq 0$ and $|B| \neq 0$

therefore $|AB| \neq 0$. Hence the matrix AB is invertible

Let us define a matrix C by the relation $C = B^{-1}A^{-1}$

Then $C(AB) = (B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$

$= B^{-1} I_n B$

$= B^{-1} B = I_n$

Also $(AB)C = (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$= A I_n A^{-1} = AA^{-1}$

$= I_n$

Thus $C(AB) = (AB)C = I_n$

Hence $C = B^{-1} A^{-1}$ is the inverse of AB.

**Elementary Row Operations and Elementary Matrices**

When we solve a system of linear algebric equations by elimination of unknown, we routinely perform three kinds of operations: Interchange of equations, multiplication of an equation by a nonzero constant and addition of a constant multiple of one equation to another equation.

When we write a homogeneous system in matrix form AX = 0, row k of A lists the coefficients in equation K of the system. The three operations on equations correspond respectively, to the interchange of two rows of A, multiplication of a row A by a constant and addition of a scalar multiple of one row of A to another row of A. We will focus on these row operations in anticipation of using them to solve the system.

**DEFINITION:** Let A be an n×n matrix. The three elementary row operations that can be performed on A are

1. Type I operation : interchanging two rows of A.
2. Type II operation : multiply a row of A by a non zero constant.
3. Type III operation : Add a scalar multiple of one row to another row.

The rows of A are m - vectors. In a type II operation, multiply a row by a non zero constant by multiplying this row vector by the number. That is, multiply each element of the row by that number. Similarly in a type III operation, we add a scalar multiple of one row vector to another row vector.

**Inverse of Non - Singular Matrices Using Elementary Transformations :**

If A is non singular matrix of order n and is reduced to the unit matrix $I_n$ by a sequence of E - row transformations only, then the same sequence of E - row transformations applied to the unit matrix $I_n$ gives the inverse of A (i.e.$A^{-1}$).

Let A be a non singular matrix of order n. It is reduced to unit matrix $I_n$ by a finite number of E- row transformations only. Here, each E - row transformation of the matrix A is equivalent to pre - multiplications by the corresponding E - matrix. Therefore, there exist elementary matrices say, $E_1, E_2, \ldots\ldots\ldots\ldots E_r$ such that

$[E_r, E_{r-1}, \ldots\ldots\ldots\ldots E_2, E_1]\ A = I_n$

Post-multiplying both sides by $A^{-1}$, we obtain

$[E_r, E_{r-1}, \ldots\ldots\ldots\ldots E_2, E_1]\ A\ A^{-1} = I_n\ A^{-1}$

$\Rightarrow [E_r, E_{r-1}, \ldots\ldots\ldots\ldots E_2, E_1]\ I_n = A^{-1}$

$\because\ AA^{-1} = I_n$

$I_n\ A^{-1} = A^{-1}$

or $A^{-1} = [E_r, E_{r-1}, \ldots\ldots\ldots\ldots E_2, E_1]\ I_n$

**Working rule to find the inverse of a non - singular matrix :**

Suppose A is a non singular matrix of order n, then first we write

$A = I_n A$

Next, we apply E row transformations to a matrix A and $I_nA$ till matrix A is reduced to $I_n$. Then B is equal to $A^{-1}$, i.e.

$B = A^{-1}$

**Example 1 :** Find by elementary row transformation the inverse of the matrix

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

(U.P.T.U. 2000, 2003)

**Solution :** The given matrix is

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

we can write the given matrix as

A= IA

Applying elementary transformations, we have

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A, R_1 \leftrightarrow R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A, R_3 \to R_3 - 3R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A, R_3 \to R_3 + 5R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A, R_3 / 2$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -15/2 & 11/2 & -3/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A, R_1 \to R_1 - 3R_3 \, R_2 \to R_2 - 3R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A, R_1 \to R_1 - 2R_2$$

i.e. $I = BA$ where $B = A^{-1}$

$$\text{or } A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} \text{Answer}$$

**Example 2 :** Find by elementary row transformations the inverse of the matrix

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

(U.P.T.U. 2002)

**Solution :** The given matrix is

$$A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

we can write the given matrix as

A = IA

$$\text{i.e. } \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying elementary transformations, we get

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A,\ R_1 \to R_1 - R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A,\ R_2 \to R_2 - 2R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -4/3 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2/3 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A,\ R_2 / -3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -4/3 \\ 0 & 0 & -1/3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2/3 & -1 & 0 \\ 2/3 & -1 & 1 \end{bmatrix} A,\ R_3 \to R_3 + R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4/3 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2/3 & -1 & 0 \\ -2 & 3 & -3 \end{bmatrix} A,\ R_3 \to R_3 \times (-3)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} A,\ R_2 \to R_2 + \frac{4}{3} R_3$$

I = BA

Where $B = A^{-1}$

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} \text{ Ans.}$$

**Normal Form :**

Every non - zero matrix of order m × n with rank r can be reduced by a sequence of elementary transformations to any of the following forms

1. $I_r$
2. $[I_r, 0]$
3. $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$
4. $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

The above forms are called normal form of A. r so obtained is a number called the rank of matrix A.

**Equivalence of matrix :**

Suppose matrix B of order m×n is obtained from matrix A (of the same order as B) by finite number of elementary transformations on A; then A is called equivalent to B i.e A~ B. Matrices A and B have same rank and can be expressed as B = PAQ, where P and Q are non singular matrices. If A is of order m×n, then P has order m×m and has n×n such that

B = PAQ

**Working rule :** Let A be a matrix of order m×n

1. we write $A = I_m A I_n$
2. Next, we transform matrix A to normal form using elementary transformations.
3. Elementary row transformation is applied simultaneously to A and $I_m$ i.e. the prefactor matrix.
4. Elementary column operation applied to A is also applied to $I_n$ i.e. the post factor matrix.
5. Finally, we find B= PAQ, where B is the normal form of A.

**Example 3 :** Reduce matrix A to its normal form, where

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$$

Hence, find the rank.

(I.A.S 2006; U.P.T.U. 2001, 2004)

**Solution :** The given matrix is

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$$

Applying elementary row transformation, we have

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix}, \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 + R_1 \end{array}$$

Now, applying elementary column transformation we have

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix}, \quad \begin{array}{l} C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 + C_1 \\ C_4 \to C_4 - 4C_1 \end{array}$$

Interchanging $C_2$ and $C_3$, we have

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & -4 \\ 0 & 4 & 0 & 0 \\ 0 & 5 & 0 & -3 \end{bmatrix}, \quad C_3 \leftrightarrow C_2$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & -4 \\ 0 & 0 & 0 & 16/5 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{array}{l} R_3 \to R_3 - \frac{4}{5} R_2 \\ R_4 \to R_4 - R_2 \end{array}$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & -4 & 0 \\ 0 & 0 & 16/5 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad C_4 \leftrightarrow C_3$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 16/5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{array}{l} R_2 \to R_2 + 4R_4 \\ R_4 \to R_4 - \frac{5}{16} R_3 \end{array}$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{matrix} R_2 \to \frac{1}{5} R_2 \\ R_3 \to \frac{5}{16} R_3 \end{matrix}$$

$$A \sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

which is the required normal form.

Here, we have three non zero rows. Thus the rank of matrix A is 3. Ans.

**Example 4 :** Find non - singular matrices P,Q so that PAQ is a normal form, where

$$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

(U.P.T.U. 2002)

**Solution :** The order of A is 3 × 4

Total number of rows in A = 3, therefore consider unit matrix $I_3$.

Total number of columns in A = 4, hence, consider unit matrix $I_4$

$\therefore\ A_{3\times 4} = I_3\, A\, I_4$

$$\begin{bmatrix} 2 & 1 & -3 & -6 \\ 2 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & -6 & -2 & -4\\ 0 & -1 & -5 & -10\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\ 0 & 1 & -3\\ 1 & 0 & -2\end{bmatrix} A \begin{bmatrix}1 & -1 & -1 & -2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \quad \begin{array}{l} C_2 \to C_2 - C_1\\ C_3 \to C_3 - C_1\\ C_4 \to C_4 - 2C_1\end{array}$$

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 5 & 10\\ 0 & 6 & 2 & 4\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\ -1 & 0 & 2\\ 0 & -1 & 3\end{bmatrix} A \begin{bmatrix}1 & -1 & -1 & -2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \quad \begin{array}{l} R_2 \to -R_2\\ R_3 \to -R_3\\ R_2 \leftrightarrow R_3\end{array}$$

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 5 & 10\\ 0 & 0 & -28 & -56\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\ -1 & 0 & 2\\ 6 & -1 & -9\end{bmatrix} A \begin{bmatrix}1 & -1 & -1 & -2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \quad \text{by } R_3 \to R_3 - 6R_2$$

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -28 & -56\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\ -1 & 0 & 2\\ 6 & -1 & -9\end{bmatrix} A \begin{bmatrix}1 & -1 & 4 & 8\\ 0 & 1 & -5 & -10\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

As by $C_3 \to C_3 - 5C_2$
& $C_4 \to C_4 - 10C_2$

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 2\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\ -1 & 0 & 2\\ -\frac{6}{28} & \frac{1}{28} & \frac{9}{28}\end{bmatrix} A \begin{bmatrix}1 & -1 & 4 & 8\\ 0 & 1 & -5 & -10\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} R_3 \to -\frac{1}{28} R_3$$

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\ -1 & 0 & 2\\ -\frac{6}{28} & \frac{1}{28} & \frac{9}{28}\end{bmatrix} A \begin{bmatrix}1 & -1 & 4 & 0\\ 0 & 1 & -5 & 0\\ 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 1\end{bmatrix} \quad \text{As by } C_4 \to C_4 - 2C_3$$

$N = PAQ$

$$P = \begin{bmatrix}0 & 0 & 1\\ -1 & 0 & 2\\ -\frac{3}{14} & \frac{1}{28} & \frac{9}{28}\end{bmatrix}, \ Q = \begin{bmatrix}1 & -1 & 4 & 0\\ 0 & 1 & -5 & 0\\ 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 1\end{bmatrix} \text{Answer}$$

**Example 5 :** Find the rank of the matrix

$$A =\sim \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$

Solution : Sometimes to determine the rank of a matrix we need not reduce it to its normal form. Certain rows or columns can easily be seen to be linearly dependent on some of the others and hence they can be reduceds zeros by E - row or column transformations. Then we try to find some non-vanishing determinant of the highest order in the matrix, the order of which determines the rank.

We have the matrix

$$A \sim \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - R_2 - R_1 \\ R_4 \to R_4 - R_3 - R_1 \end{matrix}$$

Since $\begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix} = 8 \neq 0$

Therefore rank (A) = 2 Answer

**Alter**

The determinant of order 4 formed by this matrix

$$= \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 6 & 1 & 3 & 8 \\ 6 & 1 & 3 & 8 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - R_2 \\ \& \ R_4 \to R_4 - R_3 \end{matrix}$$

$= 0$ $\quad \therefore$ $R_3$ and $R_4$ are identical.

A minor of order 3

$$= \begin{bmatrix} 6 & 1 & 3 \\ 4 & 2 & 6 \\ 10 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 3 \\ 4 & 2 & 6 \\ 6 & 1 & 3 \end{bmatrix} \text{by } R_3 \to R_3 - R_2$$

$= 0$

In similar way we can prove that all the minors of order 3 are zero.

A minor of order 2

$$\begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix} = 8 \neq 0$$

Hence rank of the matrix = 2 Answer.

**Example 6 :** Prove that the points $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ are collinear if and only if the rank of the matrix $\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$ is less than 3.

(U.P.PC.S. 1997)

**Solution :** Suppose the points $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ are collinear and they lie on the line whose equation is

$ax + by + c = 0$

Then

$ax_1 + by_1 + c = 0$ (i)

$ax_2 + by_2 + c = 0$ (ii)

$ax_3 + by_3 + c = 0$ (iii)

Eliminating a, b and c between (i), (ii) and (iii) we get

$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0$$

Thus the rank of matrix

$$A = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

is less than 3.

Conversely, if the rank of the matrix A is less than 3, then

$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0$$

Therefore the area of the triangle whose vertices are $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ is equal to zero. Hence three points are collinear.

**Consistent system of Equations:**

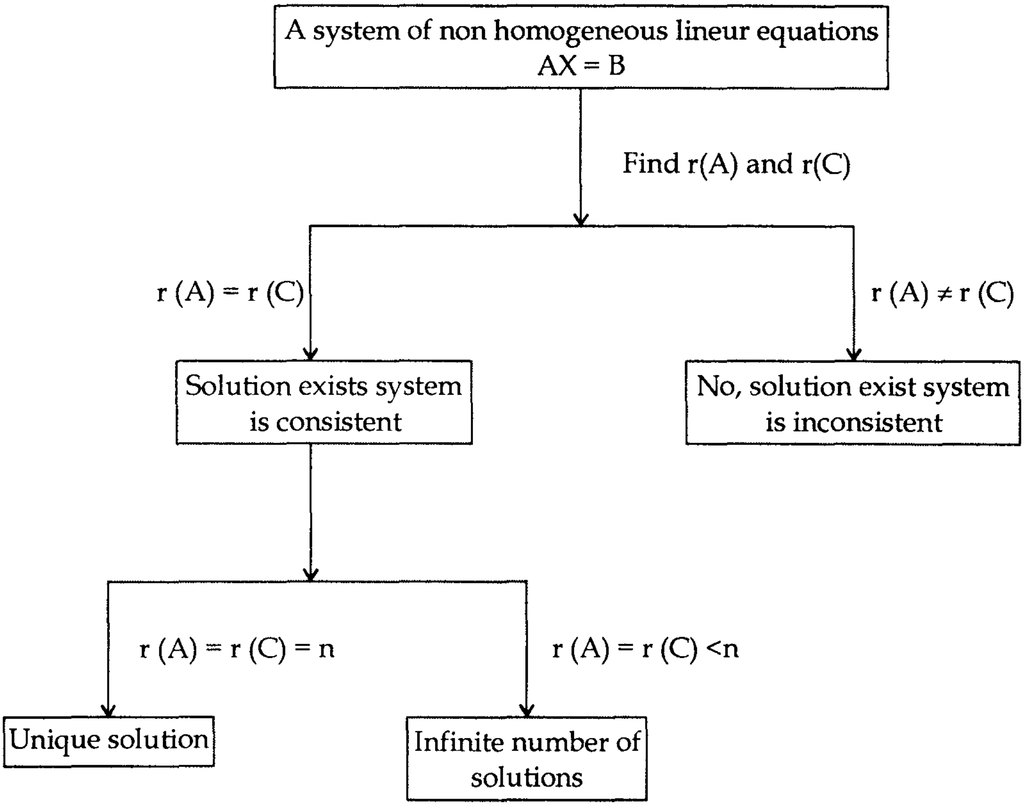
A non -homogeneous system AX = B is said to be consistent if there exists a solution. If there is no solution the system is inconsistent.

For a system of non - homogeneous linear equations AX = B (where A is the coefficient matrix) and C = [A B] is an augmented matrix :

1. If r (A) = r (C), the system is inconsistent

2. If r (A) = r (C) = n (number of unknowns) the system has a unique solution.
3. If r (A) = r (C)< n, the system has an infinite number of solutions.

The above conclusions are depicted in figure as given below


A system of non homogeneous lineur equations
AX = B
Find r(A) and r(C)
r (A) = r (C)
r (A) ≠ r (C)
Solution exists system is consistent
No, solution exist system is inconsistent
r (A) = r (C) = n
r (A) = r (C) <n
Unique solution
Infinite number of solutions

**Example 7 :** Using the matrix method, show that the equations 3x + 3y + 2z = 1; x + 2y = 4 10y + 3z = -2 2x - 3y - z = 5 are consistent and hence obtain the solution for x, y and z

(U.P.T.U. 2000)

**Solution :** The given system of linear equations can be written as
AX = B i.e.

$$\begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}$$

The augmented matrix is

$$C = [A : B] = \begin{bmatrix} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix}$$

Applying elementary row transformations to C, we have

$$C \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix}, \quad R_1 \to R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{bmatrix}, \quad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_4 \to R_4 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 0 & 29/3 & -116/3 \\ 0 & 0 & -17/3 & 68/3 \end{bmatrix}, \quad \begin{matrix} R_3 \to R_3 + \frac{10}{3} R_2 \\ R_4 \to R_4 - \frac{7}{3} R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & -4 \end{bmatrix}, \quad \begin{matrix} R_3 \to \frac{3}{29} R_3 \\ R_4 \to -\frac{3}{17} R_4 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad R_4 \to R_4 - R_3$$

Thus r(C) = r (A) = 3 hence the given system is consistant and has a unique solution

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ -4 \\ 0 \end{bmatrix}$$

or $z = -4$, $-3y + 2z = -11$, $x + 2y = 4$

or $z = -4, y = \frac{1}{3}(2z + 11) = 1, x = 4 - 2y = 2$

Thus, the solution is

$x = 2, y = 1, z = -4$ Answer.

**Example 8 :** Examine the consistency of the following system of equations and solve them if they are consistent $x_1 + 2x_2 - x_3 = 3$ ; $3x_1 - x_2 + 2x_3 = 1$ ; $2x_1 - 2x_2 + 3x_3 = 2$ ; $x_1 - x_2 + x_3 = -1$

(U.P.T.U. 2002)

**Solution :** The given system of linear equatins can be written in matrix form as A X = B i.e.

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix}$$

The augmented matrix is

C = [A : B]

$$\text{i.e. } C = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

Applying elementary row transformations to the augmented matrix, we obtain

$$C \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 5/7 & 20/7 \\ 0 & 0 & -1/7 & -4/7 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - \frac{6}{7}R_2 \\ R_4 \to R_4 - \frac{3}{7}R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_3 \to \frac{7}{5}R_3 \\ R_4 \to R_4 + \frac{1}{5}R_3 \end{matrix}$$

i.e. $r(C) = 3 = r(A)$

Hence, the system is consistent and has a unique solution, thus

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \\ 4 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + 2x_2 - x_3 = 3, - 7x_2 + 5x_3 = - 8$ and $x_3 = 4$

$\Rightarrow 7\, x_2 = 5x_3 + 8$

$\Rightarrow x_2 = \frac{1}{7}\,(5 \times 4 + 8) = 4$

and $x_1 = 3 - 2x_2 + x_3 = 3 - 8 + 4 = -1$

Thus, the solution is $x_1 = -1$ $x_2 = 4$, $x_3 = 4$ Answer

**Example 9 :** Examine the consistency of the following system of linear equations and hence, find the solution $4\,x_1 - x_2 = 12; - x_1 + 5x_2 - 2\,x_3 = 0; - 2\,x_2 + 4x_3 = - 8$

(U.P.T.U. 2005)

**Solution :** The given equations can be written in matrix form as

$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 5 & -2 \\ 0 & -2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ -8 \end{bmatrix}$$

i.e. AX = B where

$$A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 5 & -2 \\ 0 & -2 & 4 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} 12 \\ 0 \\ -8 \end{bmatrix}$$

Now the augmented matrix C is

C = [A : B]

$$\text{i.e. } C = \begin{bmatrix} 4 & -1 & 0 & 12 \\ -1 & 5 & -2 & 0 \\ 0 & -2 & 4 & -8 \end{bmatrix}$$

Applying elementary row transfomations to matrix C to reduce it to upper triangular form we get

$$B \sim \begin{bmatrix} 1 & 14 & -6 & 12 \\ -1 & 5 & -2 & 0 \\ 0 & -2 & 4 & -8 \end{bmatrix}, \qquad R_1 \to R_1 + 3\,R_2$$

$$\sim \begin{bmatrix} 1 & 14 & -6 & 12 \\ 0 & 19 & -8 & 12 \\ 0 & -2 & 4 & -8 \end{bmatrix}, \qquad R_2 \to R_2 + R_1$$

$$\sim\begin{bmatrix} 1 & 14 & -6 & 12 \\ 0 & 19 & -8 & 12 \\ 0 & 0 & 60/19 & -128/19 \end{bmatrix}, \quad R_3 \to R_3 + \frac{2}{19}R_2$$

Hence, we see that ranks of A and C are 3 i.e. r (A) = 3 = r (C). The the system of linear equations is consistent and has a unique solution. Thus, the given system of linear equations is

$$\begin{bmatrix} 1 & 14 & -6 \\ 0 & 19 & -2 \\ 0 & 0 & 60/19 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \\ -128/19 \end{bmatrix}$$

i.e. $x_1 + 14\,x_2 - 6\,x_3 = 12$

$19\,x_2 - 8x_3 = 12$

and $\frac{60}{19}x_3 = -\frac{-128}{19} \Rightarrow x_3 = \frac{-128}{60}$

i.e. $x_3 = -\frac{32}{15}$

on putting the value of $x_3$ in $19x_2 - 8\,x_3 = 12$

$$x_2 = \frac{1}{19}(12 + 8x_3) = \frac{1}{19}\left(12 + 8\left(\frac{-32}{15}\right)\right)$$

$$\Rightarrow x_2 = -\frac{4}{15}$$

Lastly, putting $x_2$, $x_3$ in $x_1 + 14\,x_2 - 6\,x_3 = 12$ we have

$$x_1 + 14 \times \left(\frac{-4}{15}\right) - 6\left(\frac{-32}{15}\right) = 12$$

$$\Rightarrow x_1 = \frac{44}{15}$$

Therefore, the solution is

$$x_1 = \frac{44}{15},\ x_2 = \frac{-4}{15},\ x_3 = \frac{-32}{15}\ \text{Answer}$$

**Example 10 :** For what values of λ and μ, the equations $x + y + z = 6$ ; $x + 2y + 3\,z = 10$; $x + 2y + \lambda z = \mu$ have (i) no solution (ii) unique solution and (iii) infinite solutions. (I.A.S 2006, U.P.T.U. 2002)

**Solution :** The given system of linear equations can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

i.e. $AX = B$

$$C = [A :] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$$

Applying elementary row transformations to C, we get

$$B \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{bmatrix}, \quad R_3 \to R_3 - R_1$$

(i) For no solution, we must have

$r(A) \neq r(C)$

i.e $\lambda - 3 = 0$ or $\lambda = 3$ and $\mu - 10 \neq 0 \Rightarrow \mu \neq 10$

(ii) for unique solution, we must have

$r(A) = r(C) = 3$

i.e. $\lambda - 3 \neq 0 \Rightarrow \lambda \neq 3$

and $\mu - 10 \neq 0 \Rightarrow \mu \neq 10$

(iii) for infinite solutions, we must have

$r(A) = r(C) < 3$

i.e. $\lambda - 3 = 0 \Rightarrow \lambda = 3$

and $\mu - 10 = 0 \Rightarrow \mu = 10$ Answer.

**Solution of Homogeneous system of Linear - Equations :**

A system of linear equations of the form AX = 0 is said to be homogeneous where A denotes the coefficient matrix and O denotes the null vector i.e.

$$a_{11} x_1 + a_{12} x_2 + \dots\dots\dots + a_{1n} x_n = 0$$
$$a_{21} x_1 + a_{22} x_2 + \dots\dots\dots + a_{2n} x_n = 0$$
$$a_{m_1} x_1 + a_{m_2} x_2 + \dots\dots\dots + a_{mn} x_n = 0$$

i.e. AX = O

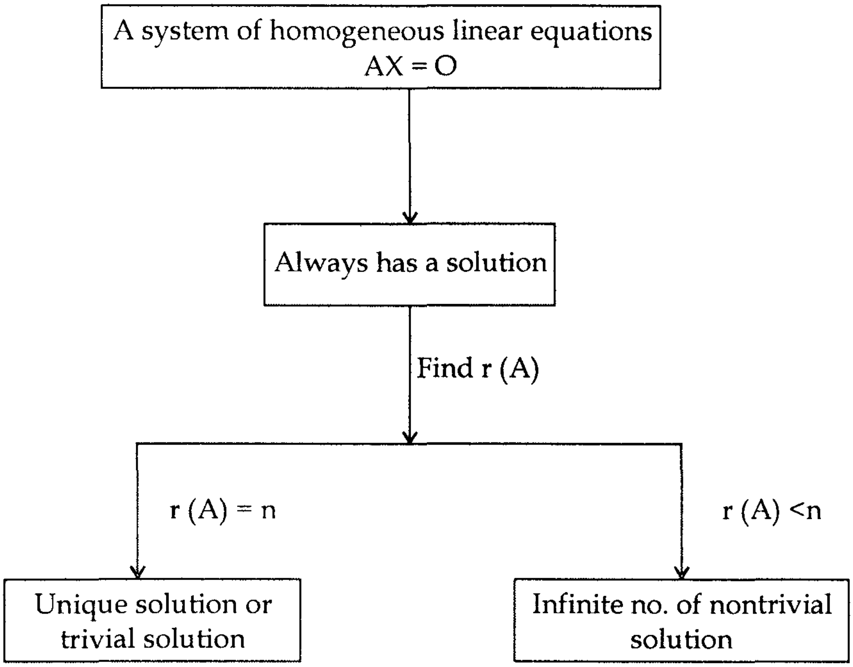
$$\text{or} \begin{bmatrix} a_{11} & a_{12} \dots\dots a_{1n} \\ a_{21} & a_{22} \dots\dots a_{21} \\ & \vdots \\ a_{m1} & a_{m2} \dots\dots a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The above system has m equations and n unknowns. We will apply the matrix method to find the solution of the above system of linear equations. For the system AX = O , we see that X = O is always a solution. This solution is called null solution or trivial solution. Thus a homogeneous system is always consistent. We will apply the techniques, already developed for non homogenous systems of linear equations to homogeneous linear equations.

(i) If r (A) = n (number of unknown) the system has only trivial solution.

(ii) If r (A) <n, the system has infinite number of solutions.

The figure as given below shows a flow chart which depicts the procedure for the solution of a homogenous system of linear equations.


A system of homogeneous linear equations
AX = O
Always has a solution
Find r (A)
r (A) = n
r (A) <n
Unique solution or trivial solution
Infinite no. of nontrivial solution

**Example 11** : Find the solution of the following homogeneous system of linear equations

$x_1 + x_2 + 2x_3 + 3x_4 = 0$; $3x_1 + 4x_2 + 7x_3 + 10x_4 = 0$

$5x_1 + 7x_2 + 11x_3 + 17x_4 = 0$; $6x_1 + 8x_2 + 13x_3 + 16x_4 = 0$

**Solution :** The given system of linear equations can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & 7 & 10 \\ 5 & 7 & 11 & 17 \\ 6 & 8 & 13 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying elementary row transformations, we get

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 5R_1 \\ R_4 \to R_4 - 6R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{array}{l} R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - R_3 \end{array}$$

i.e. $r(A) = 4 =$ number of variables

Hence the given system of homogeneous linear equations has trivial solution i.e. $x_1 = 0 = x_2 = x_3 = x_4$ Answer.

**Linear Combination of vectors :**

Let $x_1, x_2$ ................$x_k$ be a set of k vectors in $R^n$. Then the linear combination of these k vectors is sum of the form $\alpha_1 x_1 + \alpha_2 x_2 +$ ......................$+ \alpha_k x_k$, in which $\alpha_j$ is a real number.

**Linear Dependence and Independence Vectors :**

Let $x_1, x_2$ ................$x_k$ be a set of k vectors in $R^n$. Then the set is said to be linearly dependent if and only if one of the k vectors can be expressed as a linear combination of the remaining k vectors.

If the given set of vectors is not linearly dependent, it is said to be set of liearly independent vectors.

**Example 12 :** Examine for linear dependence (1, 0, 3, 1), (0, 1, -6, -1) and (0, 2, 1, 0) in $R^4$.

**Solution :** Consider the matrix equation $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$

i.e $\alpha_1(1, 0, 3, 1) + \alpha_2 (0, 1, -6, -1) + \alpha_3 (0, 2, 1, 0) = 0$

$\Rightarrow (\alpha_1 + 0\alpha_2 + 0\alpha_3, 0\alpha_1 + \alpha_2 + 2\alpha_3, 3\alpha_1 - 6\alpha_1 - 6\alpha_2 + \alpha_3, \alpha_1 - \alpha_2 + 0\alpha_3) = 0$

$\Rightarrow \alpha_1 = 0$

$\alpha_2 + 2\alpha_3 = 0$

$3\alpha_1 - 6\alpha_2 + \alpha_3 = 0$

and $\alpha_1 - \alpha_2 = 0$

$$\text{i.e} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 3 & -6 & 1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 0 & 0\\0 & 1 & 2\\0 & -6 & 1\\0 & -1 & 0\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}\qquad \begin{matrix}R_3\to R_3-3R_1\\R_4\to R_4-R_1\end{matrix}$$

$$\sim\begin{bmatrix}1 & 0 & 0\\0 & 1 & 2\\0 & 0 & 13\\0 & 0 & 2\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}\qquad \begin{matrix}R_3\to R_3+6R_2\\R_4\to R_4+R_2\end{matrix}$$

i.e. $\alpha_1 =0, \alpha_2 + 2\alpha_2 =0 \Rightarrow \alpha_2 =0$

$13\,\alpha_3 =0 \Rightarrow \alpha_3 =0$

i.e. $\alpha_1 =0 = \alpha_2 = \alpha_3$

Thus, the given vectors are linearly independent. Answer.

**Example 13 :** Examine the following vectors for linear dependence and find the relation if it exists.

$X_1 = (1, 2, 4), X_2 = (2, -1, 3), X_3 = (0, 1, 2)$ and $X_4 = (-3, 7, 2)$

(U.P.T.U. 2002)

**Solution :** The linear combination of the given vectors can be written in matrix equations as

$\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \alpha_4 X_4 = 0$

$\Rightarrow \alpha_1 (1, 2, 4) + \alpha_2 (2, -1, 3) + \alpha_3 (0, 1, 2) + \alpha_4 (-3, 7, 2) =0$

$\Rightarrow (\alpha_1 + 2\alpha_2 + 0\alpha_3 - 3\alpha_4, 2\alpha_1 - \alpha_2 + \alpha_3 + 7\alpha_4, 4\alpha_1 + 3\alpha_2 + 2\alpha_3 + 2\alpha_4) =0$

$\Rightarrow \alpha_1 + 2\alpha_2 + 0\alpha_3 - 3\alpha_4 =0$

$2\alpha_1 - \alpha_2 + \alpha_3 + 7\alpha_4 = 0$

$4\alpha_1 + 3\alpha_2 + 2\alpha_3 + 2\alpha_4 = 0$

This is a homogenous system i.e.

$$\begin{bmatrix}1 & 2 & 0 & -3\\2 & -1 & 1 & 7\\4 & 3 & 2 & 2\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\alpha_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Applying elementary row transformations we have

$$\sim\begin{bmatrix}1 & 2 & 0 & -3\\0 & -5 & 1 & 13\\0 & -5 & 2 & 14\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\alpha_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\qquad \begin{matrix}R_2\to R_2-2R_1\\R_3\to R_3-4R_1\end{matrix}$$

$$\sim\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, R_3 \rightarrow R_3 - R_2$$

Hence the given vectors are linearly independent
i.e. $\alpha_1 + 2\alpha_2 - 3\alpha_4 = 0$
$-5\alpha_2 + \alpha_3 + 13\alpha_4 = 0$
$\alpha_3 + \alpha_4 = 0$
putting $\alpha_4 = k$ in $\alpha_3 + \alpha_4 = 0$ we get $\alpha_3 = -k$
$-5\alpha_2 - k + 13k = 0$
i.e. $\alpha_2 = \frac{12}{5}k$ and $\alpha_1 + 2 \times \frac{12}{5}k - 3k = 0$
$\Rightarrow \alpha_1 = -\frac{9}{5}k$

Hence the given vectors are linearly dependent substituting the values of $\alpha$ in
$\alpha_1 X_1 + \alpha_2 X_2 + \alpha_2 X_2 + \alpha_4 X_4 = 0$ we get

$$\frac{-9k}{5}X_1 + \frac{12k}{5} \times 2 - kX_3 + kX_4 = 0$$

or $9X_1 - 12X_2 + 5X_3 - 5X_4 = 0$

**Characteristic Equation and Roots of a Matrix :**

Let $A = [a_{ij}]$ be an $n \times n$ matrix,

**(i) Characteristic matrix of A :** - The matrix $A - \lambda I$ is called the characteristic matrix of A, where I is the identity matrix.

**(ii) Characteristic polynomial of A :** The determinant $|A - \lambda I|$ is called the Characteristic polynormial of A.

**(iii) Characteristic equation of A :** The equation $|A - \lambda I| = 0$ is known as the characteristic equation of A and its roots are called the characteristic roots or latent roots or eigenvalues or characteristic values or latent values or poper values of A.

**The Cayley - Hamilton Theorem :** Every square matrix satisfies its characteristic equation i.e if for a square matrix A of order n,
$|A - \lambda I| = (-1)^n [\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots\dots\dots\dots + a_n]$
Then the matrix equation
$X^n + a_1 X^{n-1} + a_2 X^{n-2} + a_3 X^{n-3} + \dots\dots\dots\dots a_n I = 0$
is satisfied by $X = A$
i.e. $A^n + a_1 A^{n-1} + a_2 A^{n-2} + \dots\dots\dots\dots + a_1 I = 0$

(U.P.P.C.S. 2002; B.P.Sc 1997)

**Proof :** Since the element of $A - \lambda I$ are at most of the first degree in $\lambda$, the elements of Adj $(A - \lambda I)$ are ordinary polynomials in $\lambda$ of degree $n - 1$ or less.
Therefore Adj $(A - \lambda I)$ can be written as a matrix polynomial in $\lambda$, given by
Adj $(A - \lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + \ldots\ldots\ldots\ldots\ldots\ldots + B_{n-2}\lambda + B_{n-1}$ where $B_0, B_1 \ldots\ldots\ldots\ldots\ldots B_{n-1}$ are matrices of the type $n \times n$ whose elements are functions of $a_{ij}$, s
Now $(A - \lambda I)$ adj. $(A - \lambda I) = |A - \lambda I|\, I$
$\therefore$ A adj $A = |A|\, I_n$
$\therefore (A - \lambda I)(B_0\lambda^{n-1} + B_1\lambda^{n-2} + \ldots\ldots\ldots\ldots + B_{n-2}\lambda + B_{n-1})$
$= (-1)^n [\lambda^n + a_1\lambda^{n-1} + \ldots\ldots\ldots\ldots\ldots + a_n]\, I$
Comparing coefficients of like powers of $\lambda$ on both sides, we get
$- I B_0 = (-1)^n I$
$AB_0 - IB_1 = (-1)^n a_1 I$
$AB_1 - IB_2 = (-1)^n a_1 I$
……………………
……………………
$AB_{n-1} = (-1)^n a_n I$
Premultiplying these successively by $A^n, A^{n-1} \ldots\ldots\ldots\ldots\ldots\ldots I$ and adding we get
$0 = (-1)^n [A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots\ldots\ldots\ldots + a_n I]$
Thus,
$A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots\ldots\ldots\ldots\ldots + a_{n-1} A + a_n I = 0$ Proved.

**Example 14 :** Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

and verfy that it is satisfied by A and hence obtain $A^{-1}$.

(U.P.P.C.S1997; U.P.T.U. 2005)

**Solution :** We have

$$|A - \lambda I| = \begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix}$$

$= (2 - \lambda)\{(2 - \lambda)^2 - 1\} + 1\{-1(2 - \lambda) + 1\} + \{1 - (2 - \lambda)\}$
$= (2 - \lambda)(3 - 4\lambda + \lambda^2) + (\lambda - 1) + (\lambda - 1)$
$= -\lambda^3 + 6\lambda^2 - 9\lambda + 4$
we are now to verify that
$A^3 - 6A^2 + 9A - 4I = 0$ (i)
we have

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^2 = A \times A \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 \times A \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Now we can verify that $A^3 - 6A^2 + 9A - 4I$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6 \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Multiplying (i) by $A^{-1}$, we get

$A^2 - 6A + 9I - 4A^{-1} = 0$

$\therefore A^{-1} = \frac{1}{4}(A^2 - 6A + 9I)$

Now $A^2 - 6A + 9I$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + \begin{bmatrix} -12 & 6 & -6 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \quad \text{Answer.}$$

**Example 15 :** Use Cayley - Hamilton theroem to find the inverse of the following matrix

$$A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

and hence deduce the value $A^5 - 6A^4 + 6A^3 - 11A^2 + 2A + 3$

(U.P.T.U. 2002)

**Solution :** The characteristic equation of the given matrix is
$|A - \lambda I| = 0$

$$\begin{bmatrix} 4-\lambda & 3 & 1 \\ 2 & 1-\lambda & -2 \\ 1 & 2 & 1-\lambda \end{bmatrix} = 0$$

i.e. $(4-\lambda)[(1-\lambda)^2 + 4] - 3[2(1-\lambda) + 2] + 1[4 - (1-\lambda)] = 0$
or $\lambda^3 - 6\lambda^2 + 6\lambda - 11 = 0$
By Cayley - Hamilton theorem we have
$A^3 - 6A^2 + 6A - 11I = 0$
Multiplying by $A^{-1}$, we have
$A^{-1}A^3 - 6A^{-1}A^2 + 6A^{-1}A - 11A^{-1}I = 0$
or $A^2 - 6A + 6I - 11A^{-1} = 0$
$11A^{-1} = A^2 - 6A + 6I$

$$= \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} - 6 \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} + 6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix} - \begin{bmatrix} 24 & 18 & 6 \\ 12 & 6 & -12 \\ 6 & 12 & 6 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix}$$

Therefore

$$A^{-1} = \frac{1}{11} \begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix}$$

Now
$A^5 - 6A^4 + 6A^3 - 11A^2 + 2A + 3 = A^2(A^3 - 6A^2 + 6A - 11I) + 2A + 3I$
$= A^2(0) + 2A + 3I$

$$= 2\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 6 & 2 \\ 4 & 2 & -4 \\ 2 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 6 & 2 \\ 4 & 5 & -4 \\ 2 & 4 & 5 \end{bmatrix} \qquad \text{Answer.}$$

**Eigen vectors of a Matrix :**

Let A be an n × n square matrix, $\lambda$ be the scalar called eigen values of A and X be the non zero vectors, then they satisfy the equation

$AX = \lambda X$

or $[A - I\lambda]\, X = 0$

For known values of $\lambda$, one can calculate the eigen vectors.

Eigenvectors of matrices have following properties.

1. The eigen vector X of a matrix A is not unique.
2. If $\lambda_1, \lambda_2 \ldots\ldots\ldots\ldots \lambda_n$ be distinct eigen values of n × n matrix then the corresponding eigen vectors $X_1, X_2 \ldots\ldots\ldots\ldots\ldots\ldots X_n$ form a linearly independent set.
3. For two or more eigenvalues, it may or may not be possible to get linearly independent eigenvectors corresponding to the equal roots.
4. Two eigenvectors $X_1$ and $X_2$ are orthogonal if $X_1X_2 = 0$
5. Eigenvetors of a symmetric matrix corresponding to different eigenvalues are orthogonal.

In this section we will discuss four cases for finding eigenvectors, namely.

1. Eigen vectors of non-symmetric matrices with non - repeated eigenvalues.
2. Eigenvectors of non-symmetric matrix with repeated eigenvalues.
3. Eigenvectors of symmetric matrices with non - repeated eigenvalues.
4. Eigenvectors of symmetric matrices with repeated eigenvalues.

**Example 16 :** Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

(IAS 1994; U.P.P.C.S 2005; U.P.T.U. (C.O.) 2002)

**Solution :** The characteristic equation of the given matrix is

$|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

or $(3 - \lambda)(2 - \lambda)(5 - \lambda) = 0$

$\therefore \lambda = 3, 2, 5$

Thus the eigenvalues of the given matrix are

$\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 5$

The eigenvectors of the matrix A corresponding to $\lambda = 2$ is

$[A - \lambda_1 I]\, X = 0$

$$\text{i.e.} \begin{bmatrix} 3-2 & 1 & 4 \\ 0 & 2-2 & 6 \\ 0 & 0 & 5-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. $x_1 + x_2 + 4x_3 = 0$

$6x_3 = 0 \Rightarrow x_3 = 0$

$x_1 + x_2 = 0$

$\Rightarrow x_1 = -x_2 = k_1$ (say), $k_1 \neq 0$

Thus, the corresponding vector is

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ -k_1 \\ 0 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

The eigenvector corresponding to eigenvalue $\lambda_2 = 3$

$[A - \lambda_2 I]\, X = 0$

$$\begin{bmatrix} 3-3 & 1 & 4 \\ 0 & 2-3 & 6 \\ 0 & 0 & 5-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. $x_2 + 4x_3 = 0$

$-x_2 + 6x_3 = 0$

and $2x_3 = 0 \Rightarrow x_3 = 0$

i.e. $x_2 = 0 \qquad (\therefore x_3 = 0)$

Now let $x_1 = k_2$, we get the corresponding eigenvector as

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_2 \\ 0 \\ 0 \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Again when $\lambda = 5$, the eigenvector is given by

$[A - \lambda_3 I]\, X = 0$

$$\begin{bmatrix} 3-5 & 1 & 4 \\ 0 & 2-5 & 6 \\ 0 & 0 & 5-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. $-2x_1 + x_2 + 4x_3 = 0$

$-3x_2 + 6x_3 = 0$

or $x_2 = 2x_3 = k_3$ (say), $k_3 \neq 0$

Then

$2x_1 = x_2 + 4x_3 = k_3 + 2k_3$

$= 3k_3$

$$x_1 = \frac{3}{2}k_3$$

Thus, the corresponding vector is

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}k_3 \\ k_3 \\ \frac{1}{2}k_3 \end{bmatrix} = \frac{1}{2}k_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

**Example 17 :** Find the characteristic equation of the matrix

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 1 \end{bmatrix}$$

(U.P.P.C.S 2000; U.P.T.U. 2007)

Also find eigenvalues and eigenvectors of this matrix.

**Solution :**

The characteristic equation of the matrix is

$|A - \lambda I|X = 0$

$$\begin{bmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{bmatrix} = 0$$

$\Rightarrow \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

$\Rightarrow (\lambda - 1)(\lambda^2 - 4\lambda + 4) = 0$

or $\lambda = 1, 2, 2$

The eigenvectors corresponding to $\lambda_1 = 1$ is

$[A - 1I]X = 0$

$$\begin{bmatrix} 1-1 & 2 & 2 \\ 0 & 2-1 & 1 \\ -1 & 2 & 2-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_2 + 2x_3 = 0$

$x_2 + x_3 = 0 \Rightarrow x_2 = -x_3 = k_1$ (say), $K_1 \neq 0$

$-x_1 + 2x_2 + x_3 = 0$

$x_1 = 2x_2 + x_3 = 2k_1 - k_1 = k_1$

Hence, the required eigenvector is

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_1 \\ -k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Now, for the eigenvector corresponding to $\lambda_2 = 2$

$[A - 2I]X = 0$

$$\begin{bmatrix} 1-2 & 2 & 2 \\ 0 & 2-2 & 1 \\ -1 & 2 & 2-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. $-x_1 + 2x_2 + 2x_3 = 0$

$x_3 = 0$

$-x_1 + 2x_2 = 0$

$x_1 = 2x_2 = k_2$ (say), $k_2 \neq 0$

$x_1 = k_2, x_2 = \frac{1}{2}k_2, x_3 = 0$

Hence, the corresponding vector is

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_2 \\ \frac{1}{2}k_2 \\ 0 \end{bmatrix} = \frac{1}{2}k_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

**Diagonalization of Matrices :**

We have referred to the elements $a_{ii}$ of a square matrix as its main diagonal elements. All other elements are called off- diagonal elements.

**Diagonal Matrix : - Definition**

A square matrix having all off - diagonal elements equal to zero is called a diagonal matrix.

We often write a diagonal matrix having main diagonal elements $d_1$, $d_2$................$d_n$ as

$$\begin{bmatrix} d_1 & & O \\ & d_2 & \\ O & & d_n \end{bmatrix}$$

with O in the upper right and lower left corners to indicate that all off - diagonal elements are zero.

**Diagonalizable Matrix** : An n × n matrix A is diagonalizable if there exists an n × n matrix P such that $P^{-1}AP$ is a diagonal matrix.

When such P exists, we say that P is diagonalizes A.

**Example 8 :** Diagonalize the following matrix

$$A = \begin{bmatrix} -1 & 4 \\ 0 & 3 \end{bmatrix}$$

**Solution :** The eigenvalues of the given matrix are -1 and 3, and corresponding eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively

From

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Because the eigenvectors are linearly independent, this matrix is non singular (as $|P| \neq 0$), we find that

$$P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Now compute

$$P^{-1}AP = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$$

which has the eigenvalues down the main diagonal, corresponding to the order in which the eigentvectors were written as column of P.

If we use the other order in writing the eigenvectors as columns and define

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

then we get

$$Q^{-1}AQ = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

**Example 19 :** Diagonalize the matrix

$$\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(U.P.T.U. 2006)

**Solution :** The characteristic equation of the given matrix is $|A - \lambda I| = 0$ i.e.

$$\begin{bmatrix} 1-\lambda & 6 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} = 0$$

$\lambda = 3, 4, -1$

Now, eigenvectors corresponding to

$\Rightarrow \lambda = -1$ is

$[A - \lambda_1 I]\, X_1 = 0$

i.e. $$\begin{bmatrix} 1+1 & 6 & 1 \\ 1 & 2+1 & 0 \\ 0 & 0 & 3+1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 6 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. $2x_1 + 6x_2 + x_3 = 0$

$x_1 + 3x_2 = 0$

$4x_3 = 0 \Rightarrow x_3 = 0$

$x_1 = -3x_2$

suppose $x_2 = k$, then $x_1 = -3k$, $k \neq 0$

The eigenvector is

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3k \\ k \\ 0 \end{bmatrix} = k\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

Eigenvector corresponding to $\lambda_2 = 3$ is

$[A - \lambda_2 I]X_2 = 0$

$$\text{i.e.} \begin{bmatrix} 1-3 & 6 & 1 \\ 1 & 2-3 & 0 \\ 0 & 0 & 3-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 6 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. $-2x_1 + 6x_2 + x_3 = 0$

$x_1 - x_2 = 0$ $x_1 = x_2 = k$ (say), $k \neq 0$

$x_3 = 2x_1 - 6x_2 = 2k - 6k = -4k$

The eigenvector is

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ k \\ -4k \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$$

Eigenvector corresponding to $\lambda_3 = 4$ is

$[A - \lambda_3 I] X_3 = 0$

$$\begin{bmatrix} 1-4 & 6 & 1 \\ 1 & 2-4 & 0 \\ 0 & 0 & 3-4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore -3x_1 + 6x_2 + x_3 = 0$

$x_1 - 2x_2 = 0 \Rightarrow x_1 = 2x_2$

$-x_3 = 0 \Rightarrow x_3 = 0$

Let $x_2 = k$ then $x_1 = 2k$, Thus, the eigenvector is

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k \\ k \\ k \end{bmatrix} = k\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Thus, modal matrix P is

$$P = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

Now $P^{-1} = \frac{1}{20}\begin{bmatrix} 4 & -8 & -1 \\ 0 & 0 & 5 \\ -4 & -12 & -4 \end{bmatrix}$

For diagonalization

$D = P^{-1}AP$

$$= -\frac{1}{20}\begin{bmatrix} 4 & -8 & -1 \\ 0 & 0 & 5 \\ -4 & -12 & -4 \end{bmatrix}\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

$$= -\frac{1}{20}\begin{bmatrix} 4-8+0 & 24-16+0 & 4+0-3 \\ 0+0+0 & 0+0+0 & 0+0+15 \\ -4-12+0 & -24-24+0 & -4+0-12 \end{bmatrix}\begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

$$= -\frac{1}{20}\begin{bmatrix} -4 & 8 & 1 \\ 0 & 0 & 15 \\ -16 & -48 & -16 \end{bmatrix}\begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

$$= -\frac{1}{20}\begin{bmatrix} 12+8+0 & -4+8-4 & -8+8+0 \\ 0+0+0 & 0+0-60 & 0+0+0 \\ 48-48 & -16-48+64 & -32-48+0 \end{bmatrix}$$

$$= -\frac{1}{20}\begin{bmatrix} 20 & 0 & 0 \\ 0 & -60 & 0 \\ 0 & 0 & -80 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

D = dia (-1, 3, 4)  Answer.

**Complex Matrices :**

A matrix is said to be complex if its elements are complex number. For example

$$A = \begin{bmatrix} 2+3i & 4i \\ 2 & -i \end{bmatrix}$$

is a complex matrix.

**Unitary Matrix :**

A square matrix A is said to be unitary if

$A^{\theta}A = I$

Where $A^{\theta} = \left(\overline{A}\right)'$ i.e transpose of the complex conjugate matrix.

**Example 20 :** Show that the matrix

$\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary

(U.P.T.U. 2002)

**Solution :**

$$A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \text{and } A' = \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

$$\left(\overline{A}\right)' = A^{\theta} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$A^{\theta}A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \times \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{3}}\begin{bmatrix} 1.1+(1+i).(1-i) & 1(1+i)+(1+i).1 \\ (1-i).1+(-1)(1-i) & (1-i)(1+i)+(-1)(-1) \end{bmatrix}$$

$$= \frac{1}{\sqrt{3}}\begin{bmatrix} 1+1-i^2 & 0 \\ 0 & 1-i^2+1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{3}}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\Rightarrow$ A is unitary matrix. Proved

**Hermitian matrix :**

A square matrix A is said to be a Hermitian matrix if the transpose of the conjugate matrix is equal to the matrix itself i.e

$A^{\theta} = A \Rightarrow \overline{a}_{ij} = a_{ji}$

where $A = [a_{ij}]_{n\times n}$; $a_{ij} \in C$

For example

$$\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}, \begin{bmatrix} 1 & 2-3i & 3+4i \\ 2+3i & 0 & 4-5i \\ 3-4i & 4+5i & 2 \end{bmatrix}$$

are Hermitian matrices.

if A is a Hermitian matrix, then

$\bar{a}_{ii} = a_{ii} \Rightarrow \alpha - i\beta = \alpha + i\beta$

$\Rightarrow 2i\beta = 0$

$\Rightarrow \beta = 0$

$\therefore\ a_{ii} = \alpha + i\,(0)$

$a_{ii} = \alpha$

= which is purely real

Thus every diagonal element of Hermitian Matrix must be real.

(U.P.P.C.S 2005)

**Skew - Hermitian Matrix :**

A square matrix A is said to be skew - Hermitian if $A^{\theta} = -A \Rightarrow \bar{a}_{ij} = -a_{ij}$

For principal diagonal

$j = i$

$\Rightarrow \bar{a}_{ii} = -a_{ii}$

$\Rightarrow \bar{a}_{ii} + a_{ii} = 0$

$\Rightarrow$ realpart of $a_{ii} = 0$

$\Rightarrow$ diagonal elements are parely imaginary

Thus the diagonal elements of a skew - Hermitian matrix must be pure imaginary numbers or zero.

For example

$$\begin{bmatrix} 0 & -2-i \\ 2-i & 0 \end{bmatrix}, \begin{bmatrix} -i & 3+4i \\ -3+4i & 0 \end{bmatrix}$$

are Skew - Hermitian matrices.

**Problem set**

**Exercise**

1. Using elementary transformations find the inverse of matrix A where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

[U.P.T.U. (C.O.) 2007]

$$\text{Ans}.\ A^{-1} = \begin{bmatrix} -1/4 & 3/4 & 0 \\ 3/4 & -1/4 & 0 \\ -1/4 & -1/4 & 1 \end{bmatrix}$$

2. Using elementary transformations, find the inverse of the matrix A where

$$A=\begin{bmatrix}1 & 3 & 3\\ 1 & 4 & 3\\ 1 & 3 & 4\end{bmatrix}$$

$$\text{Ans. } A^{-1}=\begin{bmatrix}7 & -3 & -3\\ -1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}$$

3. Using elementary transformations, find the inverse of the matrix A where

$$A=\begin{bmatrix}i & -1 & 2i\\ 2 & 0 & 2\\ -1 & 0 & 1\end{bmatrix}$$

$$\text{Ans. }\begin{bmatrix}0 & 1/4 & -i/2\\ -1 & (3/4)i & i/2\\ 0 & 1/4 & 1/2\end{bmatrix}$$

4. Using elementary transformations, find the inverse of matrix A where

$$A=\begin{bmatrix}2 & 5 & 3 & 3\\ 2 & 3 & 3 & 4\\ 3 & 6 & 3 & 2\\ 4 & 12 & 0 & 8\end{bmatrix}$$

$$\text{Ans. } A^{-1}=\frac{1}{48}\begin{bmatrix}-144 & 36 & 60 & 21\\ 48 & -20 & -12 & -5\\ 48 & -4 & -12 & -13\\ 0 & 12 & -12 & 3\end{bmatrix}$$

5. Find the rank of the following matrix by reducing it to normal form.

(i) $$A=\begin{bmatrix}1 & 2 & -1 & 3\\ 4 & 1 & 2 & 1\\ 3 & -1 & 1 & 2\\ 1 & 2 & 0 & 1\end{bmatrix}$$

(U.P.T.U. (C.O) 2002)

(ii) $$A=\begin{bmatrix}0 & 1 & 2 & -2\\ 4 & 0 & 2 & 6\\ 2 & 1 & 3 & 1\end{bmatrix}$$

(U.P.T.U. (C.O.) 2007)

(iii) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

(U.P.T.U. 2006)

(iv) Find the rank of the matrix, $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

(U.P.T.U. 2000, 2003)

Ans. (i) 3 (ii) 2 (iii) 3 (iv) 2

6. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, find two non - singular matrices P and Q such that PAQ = I Hence find $A^{-1}$.

(U.P.T.U. 2002)

Ans. $P = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ -2 & 3 & -3 \end{bmatrix}$ and $Q \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

and $A^{-1} = QP = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

7. Find the non singular matrices P and Q such that PAQ in the normal form for the matrices below.

(i) $\begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & -3 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 1 & -2 \end{bmatrix}$

Ans. (i) $$P=\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -1 & -1 & 1\end{bmatrix},\ Q=\begin{bmatrix}1 & 1/3 & 4/3 & -1/3\\ 0 & -1/6 & -5/6 & 7/6\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

(ii) $$P=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 3/28 & 13/28 & -1/28\end{bmatrix}$$

$$Q=\begin{bmatrix}1 & 3 & -7 & 21/28\\ 0 & 1 & -2 & 10/28\\ 0 & 0 & 1 & -47/28\\ 0 & 0 & 0 & 1\end{bmatrix}$$

8. Use elementary transformation to reduce the following matrix A to upper triangular form and hence find the rank A where

$$A=\begin{bmatrix}2 & 3 & -1 & -1\\ 1 & -1 & -2 & -4\\ 3 & 1 & 3 & -2\\ 6 & 3 & 0 & -7\end{bmatrix}$$ (U.P.T.U. 2005)

Ans. 3

9. Check the consistency of the following system of linear non-homogenous equations and find the solution, if it exists.

$7x_1 + 2x_2 + 3x_3 = 16$; $2x_1 + 11x_2 + 5x_3 = 25$, $x_1 + 3x_2 + 4x_3 = 13$

(U.P.T.U. 2008)

Ans. $x_1 = \frac{95}{91},\ x_2 = \frac{100}{91},\ x_3 = \frac{197}{91}$

10. For what values of $\lambda$ and $\mu$, the following system of equations

$2x + 3y + 5z = 9$, $7x + 3y - 2z = 8$, $2x + 3y + \lambda z = \mu$

will have (i) unique solution and (ii) no solution

Ans. (i) $\lambda \neq 5$ (ii) $\mu \neq 9, \lambda = 5$

11. Determine the values of $\lambda$ and $\mu$ for which the following system of equations

$3x - 2y + z = \mu$, $5x - 8y + 9z = 3$, $2x + y + \lambda z = -1$ has

(i) Unique solution (ii) No solution and

(iii) Infinite solutions.

Ans. (i) $\lambda \neq -3$ (ii) $\lambda = -3,\ \mu \neq \frac{1}{3}$ (iii) $\lambda = -3, \mu = 1/3$

12. Verify the Cayley - Hamilton theorem for the following matrices and also find its inverse using this theorem

(i) $$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

(U.P.T.U. 2007)

(ii) $$A = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$$

(U.P.T.U. (CO) 2007)

Ans. (i) $A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$ (ii) $A^{-1} = \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 1 & -2 \\ 3 & -8 & -2 \end{bmatrix}$

13. Find the characteristic equation of the symmetric matrix

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -2 & 2 \end{bmatrix}$$

and hence also find $A^{-1}$ by Cayley - Hamilton theorem. Find the value of $A^6 - 6A^5 + 9A^4 - 2A^3 - 12A^2 - 23A - 9I$

(U.P.T.U. 2003, 2004)

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 1 \\ 0 & 3 & 3 \end{bmatrix}$$

and Value = $\begin{bmatrix} 84 & -102 & 80 \\ -80 & 106 & -80 \\ 102 & -138 & 106 \end{bmatrix}$

14. Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

verify Cayley-Hamilton theorem and hence evaluate the matrix equation
$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

(U.P.T.U. 2002)

Ans. $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

and $\begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$

15. Find the eigenvalues and corresponding eigen vectors of the following

(i) $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

(ii) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

Ans. (i) 1, 1, 7; $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(ii) 5, -3, -3; $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

(iii) 2, 2, 3, $\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ -2 \\ 5 \end{bmatrix}$ Dependent eigenvectors.

16. Diagonalize the given matrix

(i) $\begin{bmatrix} 0 & -1 \\ 4 & 3 \end{bmatrix}$

(ii) $\begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 5 & 0 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & -2 \end{bmatrix}$

(iv) $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Ans. (i) $$D = P^{-1}AP = \begin{bmatrix} \dfrac{3+\sqrt{7}i}{2} & 0 \\ 0 & \dfrac{3-\sqrt{7}i}{2} \end{bmatrix}$$

(ii) $$P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$$

(iii) $$P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

(iv) $$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & \dfrac{-5+\sqrt{5}}{2} & 0 \\ 0 & 0 & 0 & \dfrac{-5-\sqrt{5}}{2} \end{bmatrix}$$

17. Prove that $\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$ is not diagonalizable.

## OBJECTIVE PROBLEMS

Each of the following questions has four alternative answers, one of them is correct. Tick mark the correct answer.

1. The rank of the matrix $A = \begin{bmatrix} 1 & -3 & 2 \\ 3 & -9 & 6 \\ -2 & 6 & -4 \end{bmatrix}$ is

(U.P.P.C.S. 1990)

(A) 0 (B) 1
(C) 2 (D) 3

Ans. (B)

2. The equations
$x - y + 2z = 4$
$3x + y + 4z = 6$
$x + y + z = 1$ have

(A) Unique solution (B) Infinite solution
(C) No solution (D) None of these

Ans. (B)

3. If $A = \begin{bmatrix} 5 & 0 & 2 \\ 0 & 1 & 0 \\ -4 & 0 & -1 \end{bmatrix}$ and I be 3×3 unit matrix. If M = I - A, then rank of I - A is

(I.A.S. 1994)

(A) 0 (B) 1
(C) 2 (D) 3

Ans. (B)

4. If $\rho(A)$ denotes rank of a matrix A, then $\rho(AB)$ is

(I.A.S. 1994)

(A) $\rho(A)$ (B) $\rho(B)$
(C) Is less than or equal to min $[\rho(A), \rho(B)]$
(D) $> \min [\rho(A), \rho(B)]$

Ans. (C)

5. If $3x + 2y + z = 0$
$x + 4y + z = 0$
$2x + y + 4z = 0$
be a system of equations then

(I.A.S. 1994)

(A) It is inconsistent
(B) It has only the trival solution $x = 0, y = 0, z = 0$
(C) It can be reduced to a single equation and so a solution does not exist.
(D) The determinant of the matrix of coefficient is zero.

Ans. (B)

6. Consider the Assertion (A) Reason (R) given below :
Assertion (A) the system of linear equations
$x - 4y + 5z = 8$
$3x + 7y - z = 3$
$x + 15y - 11z = -14$ is inconsistent.
Reason (R) Rank $\rho(A)$ of the coefficient matrix of the system is equal to 2, which is less than the number of variables of the system.

(I.A.S. 1993)

The correct answer is -
(A) Both A and R are true and R is the correct explanation of A.
(B) Both A and R are true but R is not a correct explanation of A.
(C) A is true but R is false.
(D) A is false but R is true.

Ans. (B)

7. Consider Assertion (A) and Reason (R) given below :

Assertion (A) : The inverse of $\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$ dose not exist

Reason (R) : The matrix is non sigular.

(I.A.S. 1993)

The correct answer is ______

(A) Both A and R are true and R is the correct explanation of A.
(B) Both A and R are true but R is not a correct explanation of A.
(C) A is true but R is false.
(D) A is false but R is true.

Ans. (D)

8. If A = Diag $(\lambda_1, \lambda_2, \ldots\ldots\ldots\ldots \lambda_n)$, then the roots of the equation det (A - xI) =0 are___

(I.A.S. 1993)

(A) All equal to 1
(B) All equal to zero
(C) $\lambda_i, 1 \le i \le n$
(D) $-\lambda_i, 1 \le i \le n$

Ans. (C)

9. The matrix $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}$ then inverse matrix is given by

(A) $\begin{bmatrix} 3 & 1/2 & 1/2 \\ 4 & 3/4 & -5/4 \\ 2 & 1/4 & -3/4 \end{bmatrix}$

(B) $\begin{bmatrix} 3 & -1/2 & -1/2 \\ -4 & 3/4 & 5/4 \\ 2 & -1/4 & -3/4 \end{bmatrix}$

(C) $\begin{bmatrix} 3 & 1/2 & 1/2 \\ 4 & 3/4 & 5/4 \\ 2 & -1/4 & -3/4 \end{bmatrix}$

(D) $\begin{bmatrix} 3 & 1/2 & 1/2 \\ 4 & 3/4 & 5/4 \\ 2 & 1/4 & 3/4 \end{bmatrix}$

Ans. (B)

10. The rank of the matrix $A = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 2 & 4 \\ 3 & 1 & 2 \end{bmatrix}$ given by

(A) 0 (B) 1
(C) 1 (D) 3

Ans. (B)

11. The value of $\lambda$ for which the system of equation $x + 2y + 3z = \lambda x$, $3x + y + 2z = \lambda y$, $2x + 3y + z = \lambda x$, have non - trival solution is given by

(A) 1 (B) 2
(C) 4 (D) None of these

Ans. (D)

12. Let the matrix be $A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, which one of the following is true.

(A) $A^{-1}$ exists (B) $B^{-1}$ exist
(C) $AB = BA$ (D) None of these

Ans. (D)

13. The equations
$x + y + z = 3$
$x + 2y + 3z = 4$
$2x + 3y + 4z = 7$
have the solution -

(A) $x = 2, y = 1, z = 1$
(B) $x = 1, y = 2, z = 1$
(C) $x = 3, y = 1, z = 1$
(D) $x = 1, y = 0, z = 3$

Ans. (C)

14. If A and B are square matrices of same order, then which one of the following is true.

(A) $(AB)' = A'B'$ (B) $(AB)^{-1} = A^{-1}B^{-1}$
(C) $(A^{-1})' = (A')^{-1}$ (D) $B'AB = BA'B$

Ans. (C)

15. The inverse of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

(C) $\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$ (D) None of these

Ans. (A)

16. The rank of the matrix $\begin{bmatrix} 1 & -2 \\ -2 & 4 \\ -1 & 2 \end{bmatrix}$ is

(A) 0 (B) 1
(C) 2 (D) 3

Ans. (B)

17. Let A be an n × n matrix from the set of real numbers and $A^3 - 3A^2 + 4A - 6I = 0$
where I is n × n is unit matrix if $A^{-1}$ exists, then

(I.A.S. 1994)

(A) $A^{-1} = A - I$
(B) $A^{-1} = A + 6I$
(C) $A^{-1} = 3A - 6I$
(D) $A^{-1} = \frac{1}{6}(A^2 - 3A + 4I)$

Ans. (D)

18. Consider the following statements. Assertion (A) : If a 2 × 2 matrix, commutes with every 2×2 matrix, than it is a scalar matrix.
Reason (R) : A 2 × 2 scalar matrix commutes with every 2 × 2 matrix of these statements -

(I.A.S. 1995, 2007)

(A) Both A and R are true and R is the correct explanation of A.
(B) Both A and R are true but R is not a correct explanation of A.
(C) A is true but R is false.
(D) A is false but R is true.

Ans. (B)

19. The inverse of the matrix $\begin{bmatrix} -0.5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is

(A) $\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (B) $\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -0.25 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Ans. (D)

20. The system of equation
$x + 2y + z = 9$
$2x + y + 3z = 7$
can be expressed as

(I.A.S. 1995, 2004)

(A) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 9 \\ 7 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$

(D) None of these

Ans. (D)

21. The points $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ are collinear if the rank of the matrix.

$A = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$ is

(A) 1 or 2 (B) 2 or 3
(C) 1 or 3 (D) 2

Ans. (A)

22. To convert the Hermitian matrix into Skew Hermitian one, the Hermitian matrix, must be multiplied by

(A) $-1$ (B) $i$

(C) $-i$ (D) None of these

Ans. (B)

23. $\begin{bmatrix} 1 & i+1 & 3 \\ -2 & 1 & 5+i \\ \sqrt{2} & 3-i & 0 \end{bmatrix}$ is a 3 × 3 matrix over the set of

(A) Natural numbers (B) Integers

(C) Real numbers (D) Complex numbers

Ans. (D)

24. The system of equations

$x + 2y + 3z = 4$

$2x + 3y + 8z = 7$

$x - y + 9z = 1$ have

(A) Unique solution (B) No solution

(C) Infinite solution (D) None of these

Ans. (C)

25. A system of equation is said to be consistent if there exist..............solutions for the system -

(A) No (B) One

(C) At least one (D) Infinite

Ans. (C)

26. If the determinant of coefficient of the system of homogeneous linear equation is zero, then the system have -

(A) Trivial solution

(B) Non - trivial solution

(C) Infinite solution

(D) None of these

Ans. (B)

27. Who among the following is associated with a technique of solving a system of linear squations ?

(A) Sarrus (B) Cayley

(C) Cramer (D) Hermite

Ans. (C)

28. Matrix $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 0 & i \\ 0 & -i & 0 \end{bmatrix}$ is

(A) Unitary (B) Hermitian

(C) Skew-Hermitian (D) None of these

Ans. (B)

29. Given matrix $\begin{bmatrix} -1 & 0 & 3-i \\ 0 & 1 & 0 \\ 3+i & 0 & 0 \end{bmatrix}$ is

(A) Hermitian (B) Non - Hermitian
(C) Unitary (D) None of these

Ans. (A)

30. Matrix $\begin{bmatrix} i & 1 & 0 \\ -1 & 0 & 2i \\ 0 & 2i & 0 \end{bmatrix}$ is

(A) Hermitian (B) Skew-Hermitian
(C) Unitary (D) None of these

Ans. (B)

31. Characteristic roots of the matrix

$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix}$ is

(A) $\pm i$ (B) $\pm 1$
(C) $\pm 2$ (D) None of these

Ans. (B)

32. The eigen values of the matrix

$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ is

(A) 5, -3, -3 (B) -5, 3, 3
(C) -5, 3, -3 (D) None of these

Ans. (A)

33. The matrix A is defined by

$A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$ the eigenvalues of $A^2$ is

(A) -1, -9, -4 (B) 1, 9, 4
(C) -1, -3, 2 (D) 1, 3, -2

Ans. (B)

34. If the matrix is $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}$ then the eigenvalues of $A^3 + 5A + 8I$ are

(A) -1, 27, -8 (B) - 1, 3, -2

(C) 2, 50, -10 (D) 2, 50, 10

Ans. (C)

35. The eigenvalue of a matrix A are 1, -2, 3 the eigenvalues of $3I - 2A + A^2$ are

(A) 2, 11, 6 (B) 3, 11, 18

(C) 2, 3, 6 (D) 6, 3, 11

Ans. (A)

36. The matrix $\begin{bmatrix} 3i & 0 & 0 \\ -1 & 0 & i \\ 0 & -i & 0 \end{bmatrix}$ is

(A) Unitary (B) Hermitian

(C) Skew - Hermitian (D) None of these

Ans. (D)

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# UNIT - 4
# Multiple Integrals

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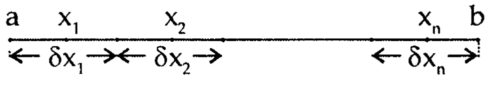
# Chapter 10

# Multiple Integrals

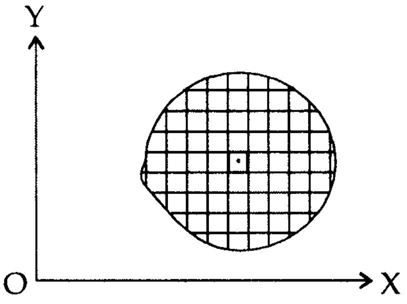
**Introduction:** The process of integration can be extended to functions of more than one variable. This leads us to two generalizations of the definite integral, namely multiple integrals and repeated integrals. Multiple integrals are definite integrals of functions of several variables. Double and triple integrals arise while evaluating quantities such as area, volume, mass, moments, centroids and moment of inertia find many applications in science and engineering problems.

**Double Integrals:** The definite integral $\int_a^b f(x)dx$ is defined as the limits of the sum $f(x_1)\,\delta x_1 + f(x_2)\,\delta x_2 + f(x_3)\,\delta x_3 + \ldots\ldots\ldots\ldots + f(x_n)\,\delta x_n$ when $n \to \propto$ and each of the length $\delta x_1$, $\delta x_2$, $\delta x_3 \ldots\ldots\ldots\ldots\ldots \delta x_n$ tends to zero. Here $\delta x_1$, $\delta x_2$, $\delta x_3 \ldots\ldots\ldots\ldots\ldots\ldots \delta x_n$ are n sub intervals into which the range b - a has been divided and $x_1$, $x_2 \ldots\ldots\ldots\ldots x_n$ are values of x lying respectively in the first, second, third,............nth sub-interval.



A double integral is its counterpart of two dimensions. Let a single valued and bounded function f(x, y) of two independent variables x, y are defined in a closed region R of the xy - plane.

Divide the region R into subregions by drawing lines parallel to Co-ordinate axes. Number of rectangles which lie entirely inside the region R, from 1 to n. Let $(x_r, y_r)$ be any point inside the rth rectangle whose area is $\delta A_r$.



Consider the sum

$f(x_1, y_1)\,\delta A + f(x_2, y_2)\,\delta A_2 + \ldots\ldots\ldots\ldots + f(x_n, y_n)\,\delta A_n$

$$=\sum_{r=1}^{n} f(x_r, y_r)\delta A_r \qquad \text{(i)}$$

Let the number of these sub-regions increase indefinitely, such that the largest linear dimension (i.e. diagonal) of $\delta A_r$ approaches zero. The limit of the sum (i), if it exists, irrespective of the mode of subdivision is called the double integral of f(x, y) over the region R and is denoted by

$$\iint_R f(x,y)\,dA$$

In other words

$$\lim_{\substack{n\to\infty \\ \delta Ar\to 0}} \sum_{r=1}^{n} f(x_r, y_r)\delta A_r = \iint_R f(x,y)\,dA$$

which is also expressed as

$$\iint_R f(x,y)\,dxdy \text{ or } \iint_R f(x,y)\,dydx$$

**Example 1:** Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2}$

(P.T.U. 2006)

**Solution:** Since the limits of y are functions of x, the integration will first be performed w.r.t y (treating x as a constant). Thus

$$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2} = \int_0^1 \left( \int_0^{\sqrt{1+x^2}} \frac{dy}{1+x^2+y^2} \right) dx$$

$$= \int_0^1 \left[ \frac{1}{\sqrt{1+x^2}} \tan^{-1}\left( \frac{y}{\sqrt{1+x^2}} \right) \right]_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \left\{ \frac{1}{\sqrt{1+x^2}} \tan^{-1}(1) \right\} dx$$

$$= \int_0^{\pi/2} \frac{1}{\sqrt{1+x^2}} \tan^{-1} \tan\frac{\pi}{4} dx$$

$$= \frac{\pi}{4} \int_0^1 \frac{dx}{\sqrt{1+x^2}}$$

$$= \frac{\pi}{4} \left[ \log\left\{ x + \sqrt{(1+x^2)} \right\} \right]_0^1 = \frac{\pi}{4} \log\left(1+\sqrt{2}\right) \qquad \text{Answer}$$

**Example 2:** Evaluate $\int_0^1 \int_0^{y^2} e^{x/y} dydx$

**Solution:** $\int_0^1 \int_0^{y^2} e^{x/y} dydx = \int_0^1 dy \int_0^{y^2} e^{x/y} dx$

$$= \int_0^1 dy \left\{ y e^{x/y} \right\}_0^{y^2}$$

Let $\frac{x}{y} = t$

$\Rightarrow dx = y\,dt$

$$= \int_0^1 y dy \left\{ e^{y^2/y} - e^0 \right\}$$

$$= \int_0^1 y dy \left( e^y - 1 \right)$$

$$= \int_0^1 y \left( e^y - 1 \right) dy$$

$$= \left\{ y\left( e^y - y \right) \right\}_0^1 - \int_0^1 1.\left( e^y - y \right) dy$$

$$= \left\{ y\left( e^y - y \right) \right\}_0^1 - \left\{ \left( e^y - \frac{y^2}{2} \right) \right\}_0^1$$

$$= \left\{ 1\left( e^1 - 1 \right) - 0 \right\} - \left\{ e^1 - \frac{1}{2} \right\} - \left\{ e^0 - 0 \right\}$$

$$= e - 1 - e + \frac{1}{2} + 1$$

$= \frac{1}{2}$ Answer.

**Example 3:** Evaluate $\int_0^a \int_0^{\sqrt{(a^2-y^2)}} \sqrt{a^2 - x^2 - y^2}\,dy dx$

(U.P.P.C.S. 2004)

**Solution:** Let the given integral be denoted by I

$$\therefore I = \int_0^a \int_0^{\sqrt{(a^2-y^2)}} \sqrt{(a^2 - x^2 - y^2)}\,dy\,dx$$

$$= \int_0^a \left[ \int_0^{\sqrt{(a^2-y^2)}} \sqrt{\left\{ (a^2 - y^2) - x^2 \right\}}\,dx \right] dy$$

$$= \int_0^a \left[ \frac{x}{2} \sqrt{\left\{ (a^2 - y^2) - x^2 \right\}} + \frac{1}{2}(a^2 - y^2)\sin^{-1}\frac{x}{\sqrt{(a^2 - y^2)}} \right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^a \left[ \frac{1}{2}\sqrt{(a^2 - y^2)}\sqrt{\left\{ (a^2 - y^2) - (a^2 - y^2) \right\}} + \frac{1}{2}(a^2 - y^2)\sin^{-1} 1 \right] dy$$

$$= \int_0^a \frac{1}{2}\left(a^2 - y^2\right)\frac{\pi}{2}dy$$

$$= \frac{\pi}{4}\int_0^a \left(a^2 - y^2\right)dy$$

$$= \frac{\pi}{4}\left[a^2y - \frac{y^3}{3}\right]_0^a$$

$$= \frac{\pi}{4}\left[a^3 - \frac{a^3}{3}\right]$$

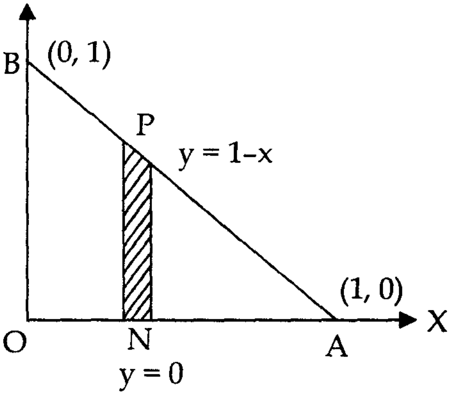
$$= \frac{\pi a^3}{6} \qquad \text{Answer.}$$

**Example 4:** Evaluate $\iint\left(x^2 + y^2\right)dxdy$ over the region in the positive quadrant for which $x + y \leq 1$

OR

Evaluate $\iint_A \left(x^2 + y^2\right)dxdy$, where A is the region bounded by $x = 0$, $y = 0$, $x + y = 1$.

**Solution:** The region of integration is the triangle OAB, for this region x varies from 0 to A i.e. from $x = 0$ to $x = 1$ and for any intermediary value of x at N. say y, varies from the x axis to P on the line AB given by $x + y = 1$ i.e. y varies from $y = 0$ to $y = 1 - x$



Hence the given integral

$$= \int_0^1 \int_0^{1-x} \left(x^2 + y^2\right)dx\,dy$$

$$= \int_0^1 \left(x^2y + \frac{y^3}{3}\right)_0^{1-x} dx$$

$$= \int_0^1 \left[ x^2(1-x) + \frac{1}{2}(1-x)^3 \right] dx$$

$$= \int_0^1 (1-x)\left\{ x^2 + \frac{1}{3}(1-x)^2 \right\} dx$$

$$= \int_0^1 \frac{1}{3}\left(1 - 3x + 6x^2 - 4x^3\right) dx$$

$$= \frac{1}{2}\left[ x - \frac{3}{2}x^2 + 2x^2 - x^4 \right]_0^1$$

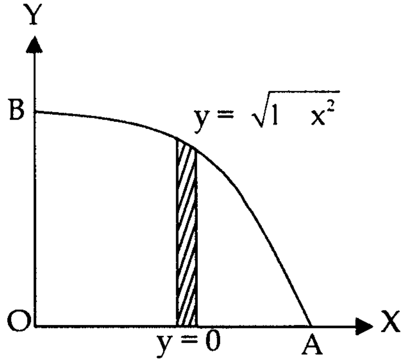
$$= \frac{1}{3}\left[ 1 - \frac{3}{2} + 2 - 1 \right]$$

$$= \frac{1}{6} \quad \text{Answer.}$$

**Example 5:** Evaluate $\iint \frac{xy}{\sqrt{(1-y^2)}}\,dx\,dy$ over the positive quadrant of the circle $x^2 + y^2 = 1$

**Solution:** The region of integration here is the quadrant OABO of the circle as shown in figure.

Here, the Co-ordinates of A and B are (1, 0) and (0, 1) respectively as the radius of the given circle is 1.



Here the given integral

$$= \int_0^1 \int_{y=0}^{\sqrt{1-x^2}} \frac{xy}{\sqrt{1-y^2}} dx\,dy$$

$$= \int_0^1 x\left[-\sqrt{(1-y^2)}\right]_0^{\sqrt{(1-x^2)}} dx$$

$$= -\int_0^1 x\left[\sqrt{\{1-(1-x^2)\}} - \sqrt{(1-0)}\right] dx$$

$$= -\int_0^1 x(x-1)dx$$

$$= -\int_0^1 (x^2-x)dx$$

$$= -\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^1$$

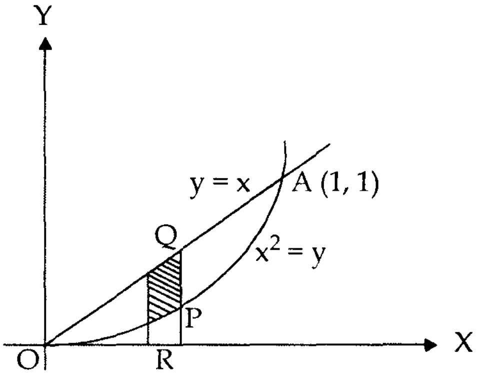
$$= -\left[\frac{1}{3} - \frac{1}{2}\right]$$

$$= \frac{1}{6} \quad \text{Answer.}$$

**Example 6:** Evaluate $\iint xy(x+y)dx\,dy$ over the area between $y = x^2$ and $y = x$

(B.P.S.C. 2005)

**Solution:** Here $x^2 = y$ represents a parabola whose vertex is the origin and axis is the axis of y. The equation $y = x$ is a line through origin making an angle of $45^0$ with x axis solving $y = x^2$ and $y = x$, we find that the parabola $y = x^2$ and the line $y = x$ intersect in the point (0, 0) and (1, 1)



Required value $= \int_{x=0}^{1} \int_{y=x}^{x^2} xy(x+y)dx\,dy$

$$=\int_{x=0}^{1}\int_{y=x}^{x^2}\left(x^2y+xy^2\right)dx\,dy$$

$$=\int_{x=0}^{1}\left[\frac{1}{2}x^2y^2+\frac{1}{3}xy^3\right]_{x}^{x^2}dx$$

$$=\int_{0}^{1}\left[\left(\frac{1}{2}x^6+\frac{1}{3}x^7\right)-\left(\frac{1}{2}x^4+\frac{1}{3}x^4\right)\right]dx$$

$$=\int_{0}^{1}\left(\frac{1}{2}x^6+\frac{1}{3}x^7-\frac{3}{6}x^4\right)dx$$

$$=\left[\left(\frac{1}{14}\right)x^7+\frac{1}{24}x^8-\frac{1}{6}x^5\right]_0^1$$

$$=\frac{1}{14}+\frac{1}{24}-\frac{1}{6}$$

$=\frac{3}{56}$ Numerically Answer.

**Example 7:** Evaluate $\iint x^2y^2\,dx\,dy$ over the region $x^2 + y^2 \le 1$

**Solution:**

$$I=\int_{-1}^{1}dx\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}x^2y^2\,dy$$

$$=\int_{-1}^{1}x^2\left(\frac{1}{3}y^3\right)_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}}$$

$$=\frac{1}{3}\int_{-1}^{1}x^2\left[\left(1-x^2\right)^{3/2}-\left\{-\left(1-x^2\right)^{3/2}\right]dx$$

$$=\frac{2}{3}\int_{-1}^{1}x^2\left(1-x^2\right)^{3/2}dx$$

$$=\frac{4}{3}\int_{0}^{1}x^2\left(1-x^2\right)^{3/2}dx$$

$$=\frac{4}{3}\int_{0}^{\pi/2}\sin^2\theta\cos^3\theta\,d\theta$$

putting $x = \sin\theta$

$$=\frac{4}{3}\int_{0}^{\pi/2}\sin^2\theta\cos^4\theta\,d\theta$$

$$=\frac{4}{3}\,\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)}{2\Gamma 4}$$

$= \frac{\pi}{24}$ Answer.

**Example 8:** Evaluate $\iint (x+y)^2 \, dx\,dy$ over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

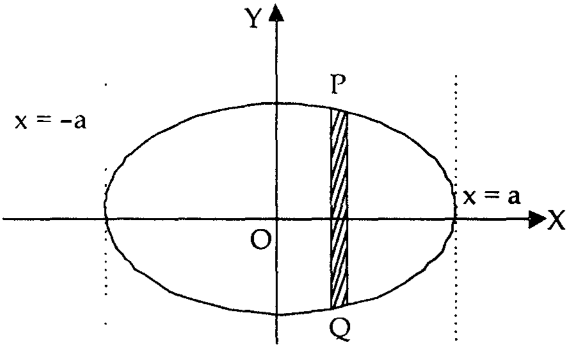
(U.P.T.U. 2004)

**Solution:** For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we have

$$y = \pm \frac{b}{a}\sqrt{(a^2 - x^2)}$$

∴ Region of integration is for x from –a to + a and for y from $-\frac{b}{a}\sqrt{(a^2 - x^2)}$ to $+\frac{b}{a}\sqrt{(a^2 - x^2)}$

∴ the given integral $\int_{x=-a}^{a} \int_{y=-\frac{b}{a}\sqrt{(a^2-x^2)}}^{\frac{b}{a}\sqrt{(a^2-x^2)}} (x+y)^2 \, dx\,dy$



$$= \int_{x=-a}^{a} \int_{y=-\frac{b}{a}\sqrt{(a^2-x^2)}}^{\frac{b}{a}\sqrt{(a^2x^2)}} \left(x^2 + y^2 + 2xy\right) dx\,dy$$

$$= 4\int_{x=0}^{a} \int_{y=0}^{\frac{b}{a}\sqrt{(a^2-x^2)}} \left(x^2 + y^2\right) dx \; dy$$

(The third integral vanishing as 2xy is an odd function of y)

$$= 4\int_{x=0}^{a}\left(x^2y+\frac{1}{3}y^3\right)_0^{\frac{b}{a}\sqrt{(a^2-x^2)}}dx$$

$$= 4\int_0^a \frac{b}{a}\sqrt{(a^2-x^2)}\left[x^2+\frac{1}{3}\frac{b^2(a^2-x^2)}{a}\right]dx$$

$$= \frac{4b}{3a^3}\int_0^a \sqrt{(a^2-x^2)}\left[3a^2x^2+b^2a^2-b^2x^2\right]dx$$

$$= \frac{4b}{3a^3}\int_{\theta=0}^{\pi/2} a\,\text{coa}\theta\left[3a^4\sin^2\theta+b^2a^2-b^2a^2\sin^2\theta\right]a\cos\theta\, d\theta$$

putting $x = a\sin\theta$

$$= \frac{4}{3}ba\int_0^{\pi/2}\left[3a^2\sin^2\theta+b^2-b^2\sin^2\theta\right]\cos^2\theta\, d\theta$$

$$= \frac{4}{3}ba\left[\left(3a^2-b^2\right)\int_0^{\pi/2}\sin^2\theta\cos^2\theta\, d\theta+b^2\int_0^{\pi/2}\cos^2\theta\, d\theta\right]$$

$$= \frac{4}{3}ab\left[\left(3a^2-b^2\right)\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma(3)}+b^2.\frac{1}{2}.\frac{\pi}{2}\right]$$

$$= \frac{4}{3}ab\left[\left(3a^2-b^2\right)\frac{\frac{1}{2}\sqrt{\pi}.\frac{1}{2}\sqrt{\pi}}{2.2}+\frac{b^2\pi}{4}\right]$$

$$= \frac{\pi ab}{3}\left[\frac{1}{4}\left(3a^2-b^2\right)+b^2\right]$$

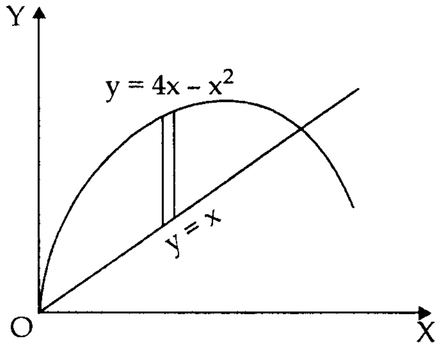
$$= \frac{1}{4}\pi ab\left(a^2+b^2\right) \qquad \text{Answer.}$$

**Example 9:** Find the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.

**Solution:** The two curves intersect at points whose abscissa are given by

$4x - x^2 = x$

i.e. $x = 0$ or $3$



The area can be considered as lying between the curve y =x, y = 4x -x², x =0 and x =3. So, integrating along a vertical strip first, we see that the required area

$$=\int_0^3\int_x^{4x-x^2} dx\,dy=\int_0^3 [y]_x^{4x-x^2} dx$$

$$=\int_0^3 \left(4x-x^2-x\right)dx$$

$$=\int_0^3 \left(3x-x^2\right)dx$$

$$=\left[\frac{3}{2}x^2-\frac{1}{3}x^3\right]_0^3$$

$$=\frac{27}{2}-9=\frac{9}{2}\quad \text{Answer.}$$

**Change to Polar Co-ordinates:**

We have x = r cosθ, y = r sinθ

Therefore

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix}$$

$$=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}$$

$$=r$$

$$\Rightarrow \iint_R f[x,y]dx\,dy=\iint_{R'} f[r\cos\theta,r\sin\theta]J\,d\theta\,dr$$

$$=\iint_{R'} f[r\cos\theta,r\sin\theta]r\,d\theta\,dr$$

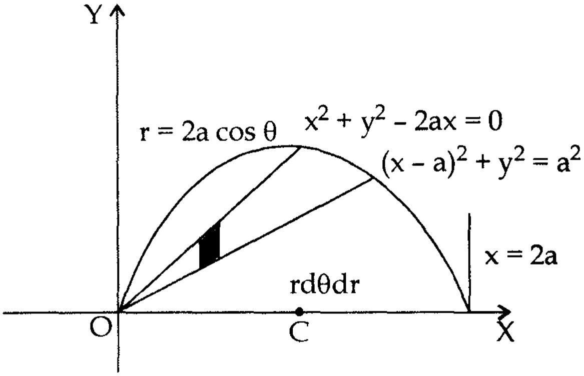
**Note:** In polar dx dy is to be replaced by r dθ dr.

**Example 10:** Evaluate $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}\left(x^2+y^2\right)dy\,dx$ by changing to polar Co-ordinates.

**Solution:** Let I = $\int_{x=}^{2a}\int_{y=0}^{\sqrt{2ax-x^2}}\left(x^2+y^2\right)dy\,dx$, upper limit of y is

$x^2 + y^2 - 2ax = 0$

$(x - a)^2 + y^2 = a^2$ (i)



This equation represent a circle whose centre is (a, 0) and radius a. Region of integration is upper half circle. Let us convert the equation into polar Co-ordinates by putting

$x = r\cos\theta$ and $y = r\sin\theta$

$\Rightarrow r^2 - 2a\,r\cos\theta = 0$

$\Rightarrow r = 2a\cos\theta$ (ii)

$$\therefore I = \int_0^{2a}\int_0^{\sqrt{(2ax-x^2)}}\left(x^2+y^2\right)dy\,dx$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} r^2\left(r\,dr\,d\theta\right)$$

$$= 4a^4\int_0^{\pi/2}\cos^4\theta\,d\theta$$

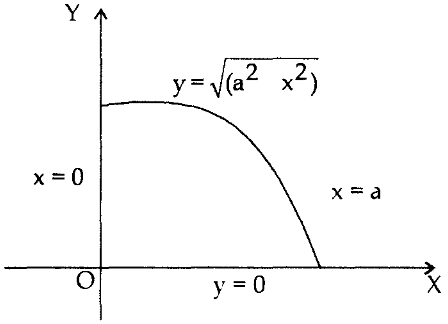
$$= \frac{3\pi a^4}{4}$$ Answer.

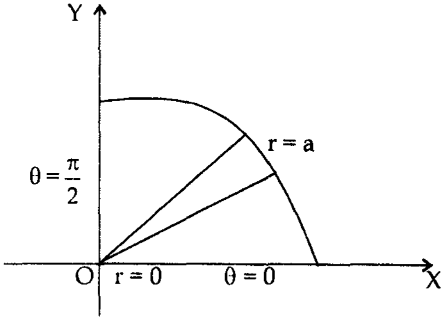
**Example 11:** Transform the integral $\int_0^a\int_0^{\sqrt{(a^2-x^2)}} y^2\sqrt{\left(x^2+y^2\right)}dx\,dy$ by changing to polar Co-ordinates and hence evaluate it.

**Solution:** The given limits of integration show that the region of integration lies between the curves

$y = 0,\ y = \sqrt{\left(a^2-x^2\right)}$, $x = 0$, $x = a$

Thus the region of integration is the part of the circle $x^2 + y^2 = a^2$ in the first quadrant. In polar Co-ordinates, the equation of the circle is
$r^2 \cos^2\theta + r^2 \sin^2\theta = a^2$
i.e. $r = a$.





Hence in polar Co-ordinates, the region of integration is bounded by the curves
$r = 0$, $r = a$, $\theta = 0$, $\theta = \pi/2$
Therefore,

$$\int_0^a \int_0^{\sqrt{(a^2-x^2)}} y^2 \sqrt{(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2} \int_0^a r^2 \sin^2\theta . r . r\, d\theta\, dr$$

$$= \int_0^{\pi/2} \sin^2\theta \left[\frac{r^5}{5}\right]_0^a d\theta$$

$$= \frac{a^5}{5} \int_0^{\pi/2} \sin^2\theta\, d\theta$$

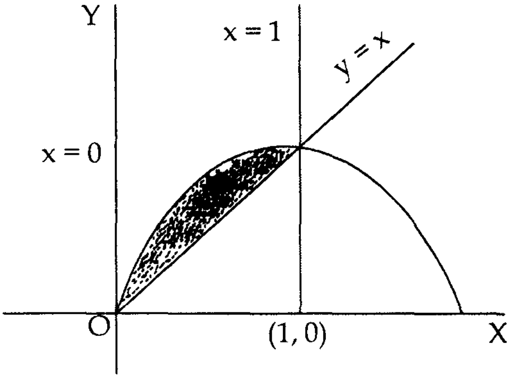
$$= \frac{a^5}{5} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(2)}$$

$$= \frac{a^5}{5}\frac{1}{2}\frac{\sqrt{\pi}.\sqrt{\pi}}{2.1} = \frac{1}{20}\pi a^5 \qquad \text{Answer}$$

**Example 12:** Evaluate $\int_0^1 \int_x^{\sqrt{(2x-x^2)}} \left(x^2+y^2\right) dx\,dy$ by changing to polars.

**Solution:** The region of integration is given by $y = x$, $y = \sqrt{\left(2x - x^2\right)}$, $x = 0$, $x = 1$.

Thus, the region of integration lies between line $y = x$, a part of circle $(x - 1)^2 + y^2 = 1$, $x = 0$ and $x = 1$.



The diameter of the circle is 2 with its end at (0, 0) and (0, 2). Its equation is r = 2 cosθ and θ varies from π/4 to $\frac{\pi}{2}$ (y =x to x =0)

Now the given integral in polar Co-ordinates takes form

$$\int_{\pi/4}^{\pi/2}\int_{0}^{2\cos\theta} r^2 r\,d\theta\,dr = \frac{1}{4}\int_{\pi/4}^{\pi/2}\left[r^4\right]_0^{2\cos\theta} d\theta$$

$$= \frac{1}{4}\int_{\pi/4}^{\pi/2} 2^4\cos^4\theta\,d\theta$$

$$= 4\int_{\pi/4}^{\pi/2}\cos^4\theta\,d\theta$$

$$= \int_{\pi/4}^{\pi/2}(1+\cos 2\theta)^2\,d\theta$$

$$= \int_{\pi/4}^{\pi/2}\left(1+2\cos 2\theta+\cos^2 2\theta\right)d\theta$$

$$= \int_{\pi/4}^{\pi/2}\left[1+2\cos 2\theta+\frac{1}{2}(1+\cos 4\theta)\right]d\theta$$

$$= \int_{\pi/4}^{\pi/2}\left[\frac{3}{2}+2\cos 2\theta+\frac{1}{2}\cos 4\theta\right]d\theta$$

$$= \left[\frac{3}{2}\theta+\sin 2\theta+\frac{1}{8}\cos 4\theta\right]_{\pi/4}^{\pi/2}$$

$$= \frac{3}{2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)+(0-1)+\frac{1}{8}(0-0)$$

$$= \frac{3\pi}{8}-1 \qquad \text{Answer.}$$

**Example 13:** Evaluate $\int_0^2\int_0^{\sqrt{2x-x^2}}\frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$ by changing to polar Co-ordinates.

**Solution:** In the given integral, y varies from 0 to $\sqrt{(2x-x^2)}$ and x varies from 0 to 2.

$y = \sqrt{2x - x^2}$

$\Rightarrow y^2 = 2x - x^2$

$\Rightarrow x^2 + y^2 = 2x$

In polar Co-ordinates, we have

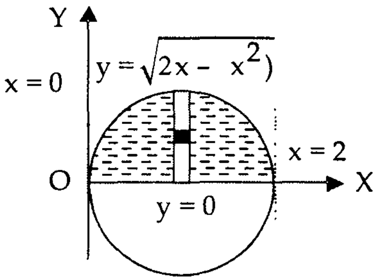
$r^2 = 2r \cos\theta$

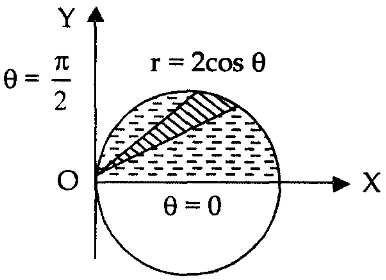
$\Rightarrow r = 2\cos\theta$

$\therefore$ For the region of integration r varies from 0 to 2 cosθ and θ varies from 0 to $\frac{\pi}{2}$.

In the given integral replacing x by r cosθ, y by r sinθ, dy dx by r dr dθ, we have

$$I = \int_0^{\pi/2}\int_0^{2\cos\theta} \frac{r\cos\theta . r\, dr\, d\theta}{r}$$





$$= \int_0^{\pi/2}\int_0^{2\cos\theta} r\cos\theta\, dr\, d\theta$$

$$= \int_0^{\pi/2} \cos\theta \left[\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta$$

$$= \int_0^{\pi/2} 2\cos^3\theta\, d\theta = 2.\frac{2}{3} = \frac{4}{3}$$ Answer

**Example 14:** Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} dx\,dy$ by changing to polar Co-ordinates. Hence show that $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

(U.P.T.U. 2002)

**Solution:** Given that

$$I = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} dx\,dy$$

Here we see that the integration is along a vertical strip extending from y =0 to y = ∞ and this strip slides from x =0 to x = ∞.

Hence, the region of integration is in the first quadrant, as shown in figure (i)

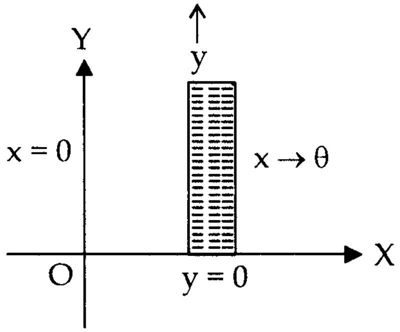
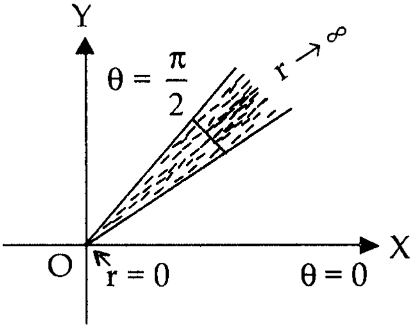


Fig. (i)

The region is covered by the radius strip from r =0 to r =∞ and it starts from θ =0 to θ = π/2 as shown in figure (ii). Thus



$$\int_0^\infty \int_0^\infty e^{-\left(x^2+y^2\right)} dx\,dy = \int_0^{\pi/2} \int_0^\infty e^{-r^2} r\,d\theta\,dr$$

$$= -\frac{1}{2}\int_0^{\pi/2} \int_0^\infty (-2r) e^{-r^2} dr\,d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2} \left[e^{-r^2}\right]_0^\infty d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2} (0-1) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 1 d\theta = \frac{\pi}{4} \qquad \text{Answer}$$

Now, let

$$I = \int_0^\infty e^{-x^2} dx \qquad (1)$$

$$\text{Also } I = \int_0^\infty e^{-y^2} dy \qquad (2)$$

(by property of definite integrals)
Multiplying (1) and (2) we get

$$I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy = \frac{\pi}{4}$$

$$\Rightarrow I = \sqrt{\left(\frac{\pi}{4}\right)} \text{ as obtained above}$$

$$\Rightarrow I = \frac{\sqrt{\pi}}{2} \qquad \text{Proved.}$$

## CHANGE OF ORDER OF INTEGRATION

**Introduction:** We have seen that $\int_a^b \int_c^d f(x,y)\,dx\,dy = \int_c^d \int_a^b f(x,y)\,dy\,dx$, provided a, b, c, d are constants.

Here we see that the limit of x and y remain the same whatever order of integrations are performed. In case, the limits are not constant the limits of x and y in both the repeated integrals will not be the same.

$$\int_a^b \int_c^d f(x,y)dx\,dy$$

Means integrate f(x,y) first w.r.t y from y = c to y = d treating x as constant and then integrate the result obtained w.r.t x from x =a to x =b.

Sometime, the evaluation of an integrated integral can be simplified by reversing the order of integration. In such cases, the limits of integration are changed if they are variable. A rough sketch of the region of integration helps in fixing the new limits of integration.

**Note:** In some books particularly American $\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x,y)dy\,dx$ is written instead of $\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x,y)\,dx\,dy$ where the first integration performed w.r.t y and then after w.r.t.x.

However in this book we shall generally use the notation given in the beginning of the introduction.

**Example 15:** Change the order of integration in the integral

$$\int_0^{a\cos\alpha} \int_{x\tan\alpha}^{\sqrt{(a^2-x^2)}} f(x,y)dx\,dy$$

**Solution:** The given integral $\int_0^{a\cos\alpha} \int_{x\tan\alpha}^{\sqrt{(a^2-x^2)}} f(x,y)dx\,dy$

Here the limits are given by x =0, x = a cosα; y = x tanα, $y = \sqrt{(a^2-x^2)}$,

$y = \sqrt{(a^2-x^2)}$ gives $x^2 + y^2 = a^2$ i.e. circle with centre at origin.

To find intersection point of

$x^2 + y^2 = a^2$ (i)

and

$y = x \tan \alpha$ (ii)

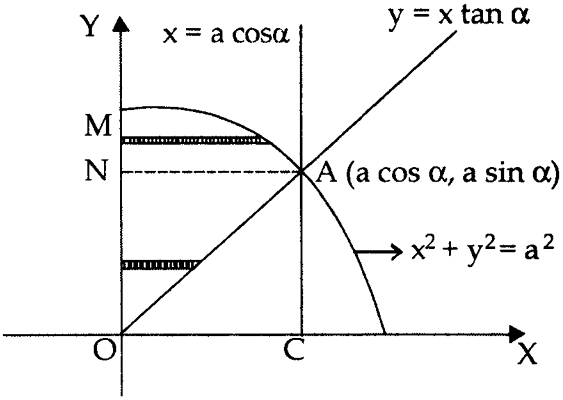
Now from (i) and (ii) $\Rightarrow x^2 + x^2 \tan^2 a = a^2$

$\Rightarrow x = a \cos \alpha$

put this in (ii), $y = a \cos \alpha \tan \alpha$

$= a \sin \alpha$

$\therefore$ $(a \sin \alpha, a \cos \alpha)$ is the intersection point A of (i) and (ii)



Through A draw a line AN parallel to x-axis. Evidently the region of integration is OAMO.

In order to change the order of integration let us take elementary strips parallel to x-axis. Such type of strips change their character at the point A. Hence the region of integration is divided into two parts ONAO, NAMN.

In the region ONAO, the strip has its extermities on the lines $x = 0$ and $y = x \tan \alpha$.

$\therefore$ limits of x in term of y are from $x = 0$ to $\frac{y}{\tan \alpha}$ (or $y \cot \alpha$) limits of y are form 0 to $a \sin \alpha$.

In the region NAMN, the strip has its extremities on the line $x = 0$ and the circle $x_2 + y^2 = a^2$

$\therefore$ limits of x in term of y are from 0 to $\sqrt{(a^2 - x^2)}$ and limits of y are from $a \sin \alpha$ to a.

$$\therefore \int_0^{a\cos\alpha} \int_{x\tan\alpha}^{\sqrt{(a^2-x^2)}} f(x,y)\,dx\,dy = \int_0^{a\sin\alpha} \int_0^{y\cot\alpha} f(x,y)\,dy\,dx + \int_{a\sin\alpha}^{a} \int_0^{\sqrt{(a^2-y^2)}} f(x,y)\,dy\,dx$$

Answer.

**Example 16:** Change the order of integration in the following integrals.

(i) $\int_0^{2a}\int_{x^2/4a}^{3a-x}\phi(x,y)dxdy$

(ii) $\int_0^{a}\int_0^{\sqrt{(a^2-x^2)}}f(x,y)dxdy.$

**Solution:** (i) We denote the given integral by I. Here the limits are given by x =0, x =2a; y = 3a -x

or $\frac{x}{3a}+\frac{y}{3a}=1$, $y=\frac{x^2}{4a}$ or $x^2 = 4ay$

For intersection point of

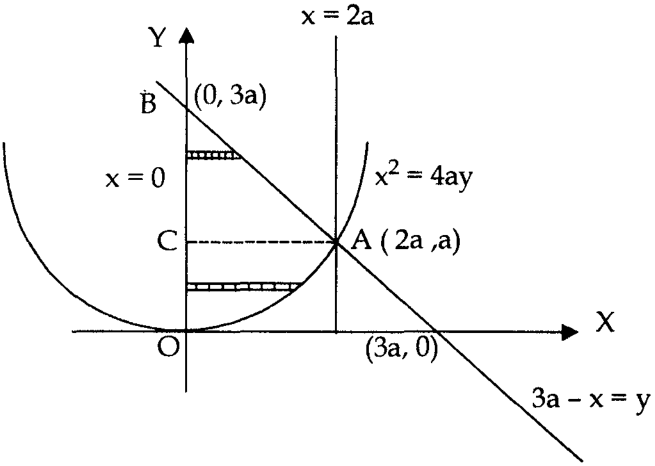
y =3a -x and $x^2 = 4ay$ we have $x^2 = 4a (3a -x)$

or (x + 6a) (x -2a) =0

or x = 2a, - 6a

Put in y = 3a -x we get y = a, 9a

∴ (-6a, 9a) (2a, a)



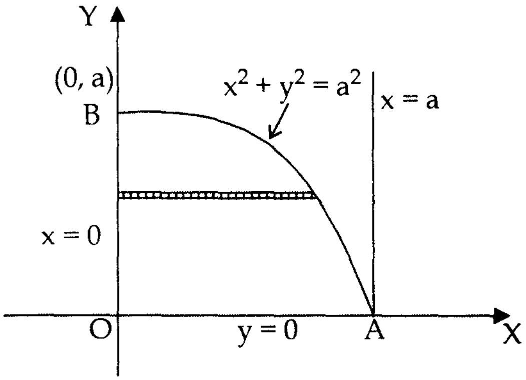
Range of integration is OABO. In order to change the order of integration, we take elementary strip parallel to x axis, such type of strips change their character at A. Hence we draw line CA parallel to x axis. Thus the range is divided into two parts OACO and CABC.

In the region OACO, any strip parallel to x axis has its extremities on x =0 and $x^2 = 4ay$. ∴ For such strips, limits of x in term of y are from 0 to $\sqrt{(4ay)}$ and limits of y are from o to a.

In region CABC, any strip parallel to x axis has its extermities on x =0 and x = 3a-y and y varies from y =a to y = 3a.

$$I=\int_0^{a}\int_0^{\sqrt{(4ay)}}\phi(x,y)dydx+\int_0^{3a}\int_0^{3a-y}\phi(x,y)dydx \qquad \text{Answer.}$$

(ii) Limits x =0, x =a; y =0, $y = \sqrt{(a^2 - x^2)}$ or $x^2 + y^2 = a^2$



Range is OABO.

∴ on changing the order of integration, we get

$$I = \int_0^a \int_0^{\sqrt{(a^2-y^2)}} f(x,y)\,dy\,dx \qquad \text{Answer.}$$

**Example 17:** Change the order of integration of

$$\int_0^{2a} \int_{\sqrt{(2ax-x^2)}}^{\sqrt{2ax}} f(x,y)\,dx\,dy$$

**Solution:** Limits of integration are x =0, x =2a, $y = \sqrt{(2ax - x^2)}, y = \sqrt{(2ax)}$

$y = \sqrt{2ax} \Rightarrow y^2 = 2ax$, parabola

$y = \sqrt{2ax - x^2} \Rightarrow x^2 + y^2 - 2ax = 0$

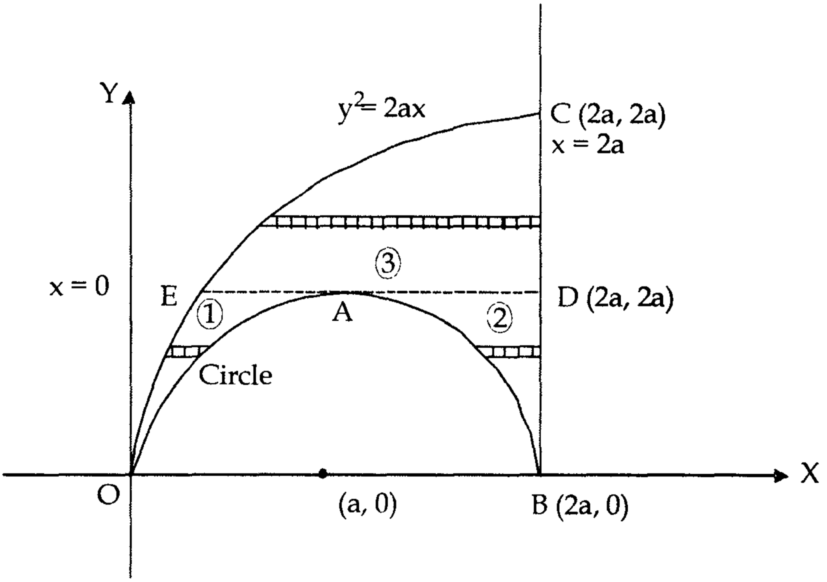
$\Rightarrow (x - a)^2 + (y - 0)^2 = a^2$, is circle with centre at (a, 0) and the radius a

$\Rightarrow x = \pm a \pm \sqrt{(a^2 - y^2)}$

intersection of x = 2a and $y^2 = 2ax$ is C (2a, 2a).

Range of integration is OABCO. Through A draw a line ED parallel to x axis. Thus the range is divided in three parts namely.

(1) OAEO
(2) ABDA
(3) EDCE



Range No (1) for the region OAEO strip parallel to x axis lies its one end on $x = y^2/2a$ and the other end on $x = a - \sqrt{(a^2 - y^2)}$. For this strip y varies from y =0 to y =a.

Range No (2) for the region ABDA, the limits for x are from $a + \sqrt{(a^2 - y^2)}$ to 2a and that for y are from y =0 to y =a.

Range No (3), for the region EDCE, the limits for x are from $y^2/2a$ (from parabola) to 2a and that for y are from y =a to y = 2a.

Hence the transformed integral is given by

$$\int_0^{2a}\int_{\sqrt{(2ax-x^2)}}^{\sqrt{(2ax)}} f(xy)\,dx\,dy = \int_0^{a}\int_{y^2/2a}^{a-\sqrt{(a^2-y^2)}} f\,dy\,dx + \int_0^{a}\int_{a+\sqrt{(a^2-y^2)}}^{2a} f\,dy\,dx + \int_0^{2a}\int_{y^2/2a}^{2a} f\,dy\,dx$$

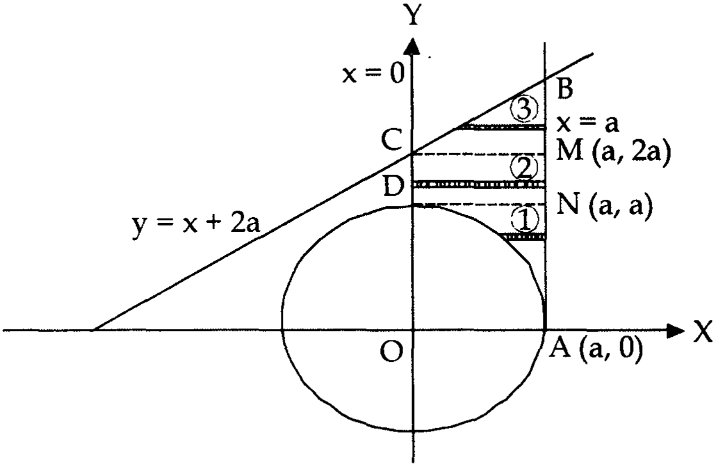
Answer.

**Example 18:** Change the order of integration in

$$\int_0^{a}\int_{\sqrt{(a^2-x^2)}}^{x+2a} \phi(x,y)\,dx\,dy$$

**Solution:** We denote the given integral by I. Limits of integration are x =0, x =a; $y = \sqrt{(a^2 - x^2)}$, y =x + 2a, $y = \sqrt{(a^2 - x^2)}$ gives $x^2 + y^2 = a^2$ circle. y = x + 2a is expressible as

$$\frac{x}{(-2a)} + \frac{y}{2a} = 1$$



The range is ABCDA. Any strip parallel to x-axis change its character at D and C both. Hence we draw two parallel lines DN and CM. The range is divided in three parts namely (1) DAND (2) DNMCD, (3) CMBC

Range (1) one end of the strip lies on $x=\sqrt{(a^2-y^2)}$ and the other end on x = a. For this strip y varies from y =0 to y =a. Similar calculations are done for range (2) and (3)

$$\therefore I=\int_0^a\int_{\sqrt{(a^2-y^2)}}^a \phi\, dy\, dx+\int_a^{2a}\int_0^a \phi\, dy\, dx+\int_{2a}^{3a}\int_{y-2a}^a \phi\, dy\, dx \quad \text{Answer.}$$

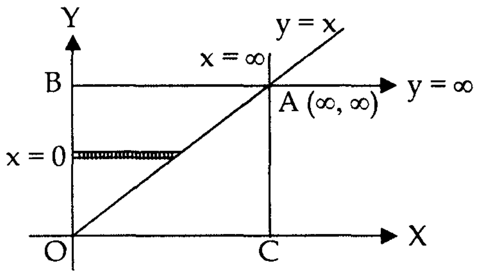
**Example 19:** Change the order of integration in $\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}dx\,dy$ and hence find its value.

(I.A.S. 2006)

**Solution:** Let $I=\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}dx\,dy$

Here the limits are

x =0, x = ∞; y =x, y = ∞



The range of integration is OABO. In order to change the order of integration, let us take an elementary strip parallel to x axis. One end of this strip is on x =0 and the other on x =y. For this strip y varies from y = 0 to y = ∞

$$\therefore \int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dx\,dy = \int_0^\infty \int_0^y \frac{e^{-y}}{y} dy\,dx$$

$$= \int_0^\infty \frac{e^{-y}}{y} dy \int_0^y dx$$

$$= \int_0^\infty \frac{e^{-y}}{y} dy [x]_0^y$$

$$= \int_0^\infty \frac{e^{-y}}{y} dy.y$$

$$= \int_0^\infty e^{-y} dy$$

$$= \left[-e^{-y}\right]_{y=0}^\infty$$

$= 1-0 = 1$ Answer.

**Example 20:** Change the order of integration in $\int_0^{a/2}\int_{x^2/a}^{x-x^2/a} f(x,y)\,dx\,dy$

**Solution:** The limits of integration are given by the parabolas i.e. $x^2/a = y$; i.e. $x^2 = ay$; $x-x^2/a = y$ i.e. $ax - x^2 = ay$ and the lines x =0; $x = \frac{a}{2}$.

Also the equation of parabola $ax - x^2 = ay$ may be written as $\left(x-\frac{a}{2}\right)^2 = -a\left(y-\frac{a}{4}\right)$ i.e. this parabola has the vertex as the point $\left(\frac{a}{2},\frac{a}{4}\right)$ and its concavity is downwards.

The points of intersection of two parabolas are given as follows $ax - x^2 = x^2$ or x= 0, $\frac{a}{2}$ and hence from $x^2 = ay$, we get y = 0 at x =0 and $y = \frac{a}{4}$ at $x = \frac{a}{2}$.

Hence the points of intersection of the two parabolas are (0,0) $\left(\frac{a}{2},\frac{a}{4}\right)$.

Draw the two parabolas $x^2 = ay$ and $ax = x^2 - ay$ intersecting at the point O (0,0) and $P\left(\frac{a}{2},\frac{a}{4}\right)$.

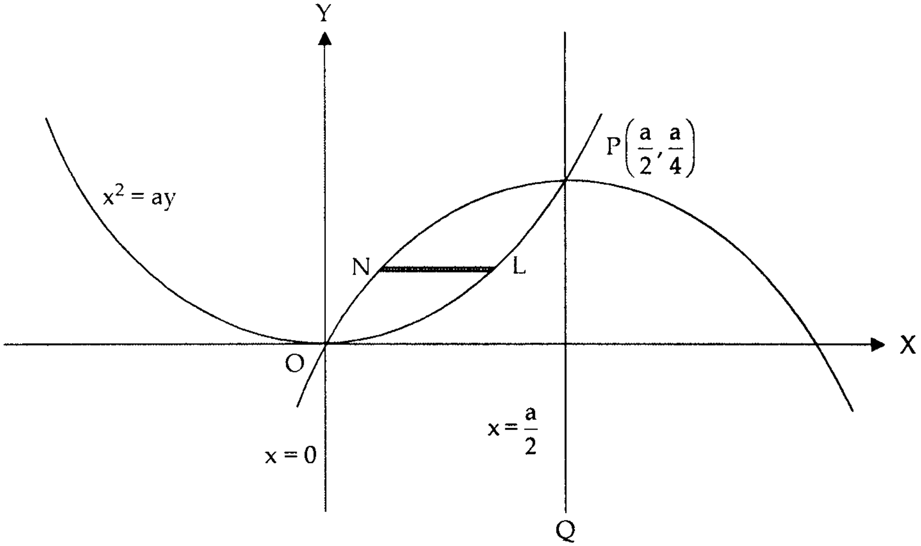
Now draw the lines x =0 and $x = \frac{a}{2}$. Clearly the integral extends to the area ONPLO.

Now take strips of the type NL parallel to the x axis.
Solving ay = ax – x² i.e x² - ax + xy =0 for x, we get

$$x=\frac{1}{2}\left[a\pm\sqrt{\left(a^2-4ay\right)}\right]$$

$$=\frac{1}{2}\left[a-\sqrt{\left(a^2-4ay\right)}\right] \qquad (1)$$

rejecting the positive sign before square root, Since x is not greater than $\frac{a}{2}$ for the region of integration.



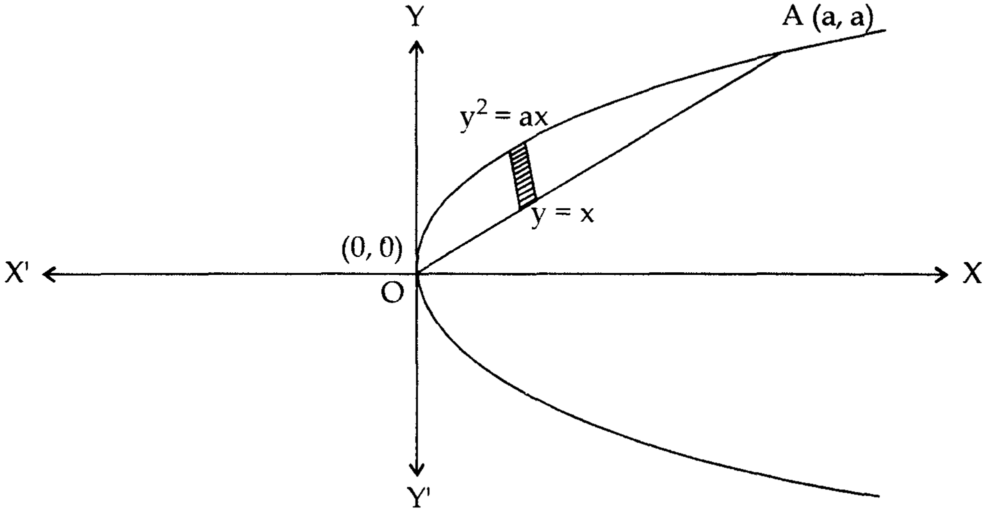
Again the region ONPLO, the elementary strip NL has the extermities N and L on ax - x = ay and x² = ay. Thus the limits of x are from $\frac{1}{2}\left[a-\sqrt{\left(a^2-4ay\right)}\right]$ to $\sqrt{(ay)}$. For limits of y, at 0, y = 0 and at P, $y=\frac{a}{4}$. Hence changing the order of integration, we have $\int_0^{a/2}\int_{x^2/a}^{x-x^2/a} f(x,y)\,dx\,dy=\int_0^{a/4}\int_{\frac{1}{2}\left[a-\sqrt{\left(a^2-4ay\right)}\right]}^{\sqrt{(ay)}} f(x,y)\,dy\,dx$ Answer.

**Example 21:** By changing the order of integration, evaluate

$$\int_0^a\int_{y^2/a}^{y}\frac{y}{(a-x)\sqrt{ax-y^2}}\,dx\,dy$$

(I.A.S. 2003, U.P.T.U. 2002)

**Solution:** The given limits show that the area of integration lies between x = y²/a, x =y, y =0, y =a since x = y²/a, y² = ax (a parabola) and y = x is a straight line, these two intersect each other in the point 0 (0, 0) and A (a, a). The area of integration is the shaded portion in the figure.



We can consider it as lying between y = x, $y = \sqrt{ax}$ ; x =0, x =a.

Therefore by changing the order of integration we have

$$\int_0^a \int_{y^2/a}^{y} \frac{y}{(a-x)\sqrt{ax-y^2}}\,dx\,dy = \int_0^a \int_x^{\sqrt{ax}} \frac{y\,dy\,dx}{(a-x)\sqrt{ax-y^2}}$$

$$= \int_0^a \left[ -\frac{(ax-y^2)^{1/2}}{a-x} \right]_x^{\sqrt{ax}} dx$$

$$= \int_0^a \frac{(ax-x^2)^{1/2}}{a-x} dx$$

$$= \int_0^a \left( \frac{x}{a-x} \right)^{1/2} dx$$

put x =a $\sin^2\theta$

$\therefore$ dx = 2a sinθ cosθ dθ

$$= \int_0^{\pi/2} \left( \frac{a\sin^2\theta}{a\cos^2\theta} \right)^{1/2} .\ 2a \sin\theta \cos\theta\, d\theta$$

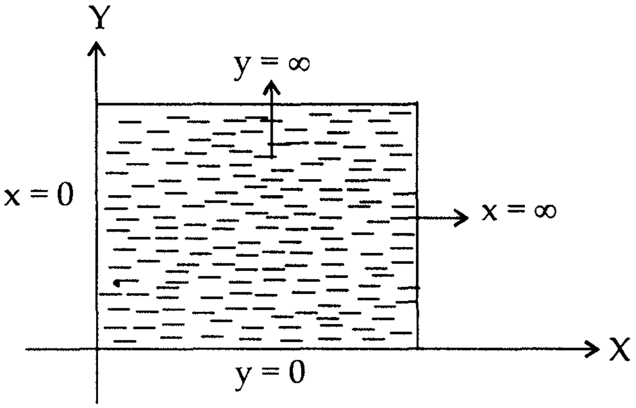
$$= \int_0^{\pi/2} 2a\sin^2\theta\, d\theta = 2a.\frac{1}{2}.\frac{\pi}{2}$$

$= \dfrac{\pi a}{2}$ Answer.

**Example 22:** Changing the order of integration of $\int_0^\infty\int_0^\infty e^{-xy}\sin nx\,dx\,dy$ show that $\int_0^\infty \frac{\sin nx}{x}dx=\frac{\pi}{2}$

(U.P.T.U 2003, 2009)

**Solution:** The region of integration is bounded by $x=0$, $x=\infty$, $y=0$, $y=\infty$.
i.e., the first quadrant, as shown in figure.



Thus

$$\int_0^\infty\int_0^\infty e^{-xy}\sin nx\,dx\,dy=\int_0^\infty dy\int_0^\infty e^{-xy}\sin nx\,dx$$

$$=\int_0^\infty\left[\frac{e^{-xy}}{n^2+y^2}\{-y\sin nx-n\cos nx\}\right]_0^\infty dy$$

$$=\int_0^\infty\left[0+\frac{n}{n^2+y^2}\right]dy$$

$$=\left[\tan^{-1}\frac{y}{n}\right]_0^\infty$$

$$=\frac{\pi}{2} \qquad \text{(i)}$$

on changing the order of integration, we get

$$\int_0^\infty\int_0^\infty e^{-xv}\sin nx\,dx\,dy=\int_0^\infty \sin nx\,dx\int_0^\infty e^{-xv}dy$$

$$= \int_0^{\infty} \sin nx\, dx \left[ \frac{e^{-xy}}{-x} \right]_0^{\infty}$$

$$= \int_0^{\infty} \frac{\sin nx}{x} dx \left[ -\frac{1}{e^{xy}} \right]_0^{\infty}$$

$$= \int_0^{\infty} \frac{\sin nx}{x} dx[-0+1]$$

$$= \int_0^{\infty} \frac{\sin nx}{x} dx \qquad \text{(ii)}$$

Hence from equations (i) and (ii) we have

$$\int_0^{\infty} \frac{\sin nx}{x} = \frac{\pi}{2}$$

**Example 23:** Find the area enclosed between the parabola $y = 4x - x^2$ and the line $y = x$.

(U.P.T.U. 2008)

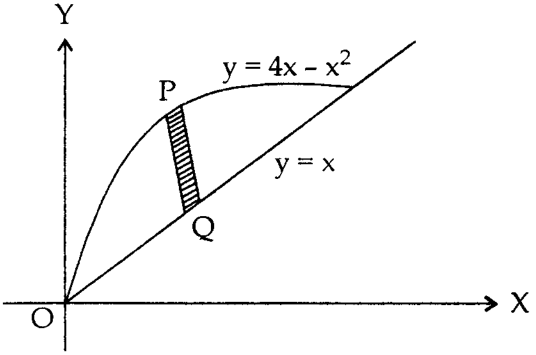
**Solution:** The given curves intersect at the points whose abscissas are given by $y = 4x - x^2$ and $y = x$, Therefore

$x = 4x - x^2$

or $3x - x^2 = 0$

$\Rightarrow x(3 - x) = 0$

$\Rightarrow x = 0, 3$



The area under consideration lies between the curves $y = x$, $y = 4x - x^2$, $x = 0$ and $x = 3$.

Hence, integrating along the vertical strip PQ first, we get the required area as

$$\text{Area} = \int_0^3 \int_{y=x}^{y=4x-x^2} dy\, dx$$

$$= \int_0^3 [y]_x^{4x-x^2} dx$$

$$= \int_0^3 \left(4x - x^2 - x\right) dx$$

$$= \int_0^3 \left(3x - x^2\right) dx$$

$$= \left[\frac{3}{2}x^2 - \frac{x^3}{3}\right]_0^3$$

$$= \left[\frac{3}{2}(3)^2 - \frac{(3)^3}{2}\right]$$

$$= \frac{27}{2} - 9 = \frac{9}{2} \quad \text{Answer.}$$

**Example 24:** By double integration, find the whole area of the curve $a^2 x^2 = y^3 (2a - y)$

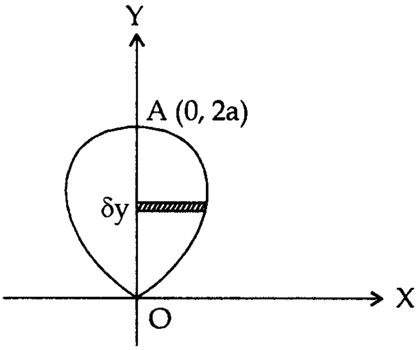
(U.P.T.U. 2001)

**Solution:** The region of integration is shown in figure. Here, we have

Area = 2 × area of the region OAB

$$= 2\int_{y=0}^{2a} \int_{x=0}^{f(y)} dx\, dy$$

where $f(y) = \dfrac{y^{3/2}\sqrt{(2a - y)}}{a}$



Consider the horizontal strip PQ with a small area, we get

$$\text{Area} = = 2\int_0^{2a} \int_0^{y^{3/2} \frac{\sqrt{2a-y}}{a}} dx\, dy$$

$$= 2\int_0^{2a} [x]_0^{y^{3/2} \frac{\sqrt{(2a-y)}}{a}} dy$$

$$= \frac{2}{a}\int_0^{2a} y^{3/2} \sqrt{2a - y}\, dy$$

putting $y = 2a\sin^2\theta$

i.e. $dy = 4a\sin\theta\cos\theta\, d\theta$ we get

$$\text{Area} = \frac{2}{a}\int_0^{\pi/2}\left(2a\sin^2\theta\right)^{3/2}\sqrt{\left(2a - 2a\sin^2\theta\right)}\; 4a\sin\theta\cos\theta\, d\theta$$

$$= 32a^2\int_0^{\pi/2}\sin^4\theta\cos^2\theta\, d\theta$$

$$= 32a^2\,\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma(4)}$$

$$= 16a^2\,\frac{\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}\times\frac{1}{2}\sqrt{\pi}}{6}$$

$= \pi a^2$ Answer.

**EXERCISE**

1. Evaluate the integral by changing the order of integration

$$\int_0^{\infty}\int_0^{x} x e^{-x^2/y}\, dy\, dx$$

(U.P.T.U. 2006)

Ans. $\frac{1}{2}$

2. Evaluate

$$\int_0^1 dx\int_0^x e^{y/x}\, dy$$

Ans. $\frac{1}{2}$ (e - 1).

3. Evaluate $\iint_R xy\, dx\, dy$ where R is the quadrant of the circle $x^2 + y^2 = a^2$ where $x \geq 0$ and $y \geq 0$

Ans. $\frac{a^4}{8}$

4. Evaluate the following integral by changing the order of integration

$$\int_0^1\int_{e^x}^{e}\frac{dy\, dx}{\log y}$$

Ans. $e^{-1}$

5. Evaluate by changing the order of integration

$$\int_0^1\int_{2y}^{2} e^{x^2}\, dx\, dy$$

Ans. $\frac{e^4 - 1}{4}$

6. Evaluate $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z\,dz\,dx\,dy$

Ans. $\frac{1}{4}(e^2 - 8e + 13)$

7. Change the order of integration in

$$\int_0^a \int_x^{a^2/x} (x+y)\,dx\,dy$$

and find its value.

Ans. $\int_0^a \int_x^{a^2/x} (x+y)\,dx\,dy = \int_0^a \int_0^y (x+y)\,dy\,dx + \int_a^\infty \int_0^{a^2/y} (x+y)\,dy\,dx$ and its value is $\infty$

8. Change the order of integration of the integral

$$\int_0^a \int_0^{b/(b+x)} f(x,y)\,dx\,dy$$

Ans. $\int_0^{b/a+b} \int_0^a f(x,y)\,dy\,dx + \int_{b/a+b}^1 \int_0^{b(1-y)/y} f(x,y)\,dy\,dx$

9. Transform $\int_0^{\pi/2} \int_0^{\pi/2} \sqrt{\frac{\sin\phi}{\sin\theta}}\,d\phi\,d\theta$ by the substitution $x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$ and show that its value is $\pi$.

(U.P.T.U. 2001)

10. Let D be the region in the first quadrant bounded by $x = 0$, $y = 0$ and $x + y = 1$, change the variables x, y to u,v where $x + y = u$, $y = uv$ and evaluate $\iint_D xy(1-x-y)^{1/2}\,dx\,dy$

(U.P.T.U. 2002)

Ans. $\frac{16}{945}$

11. Determine the area of the region bounded by the curves $xy = 2$, $4y = x^2$, $y = 4$.

Ans. $\frac{28}{3} - 4\log 2$

12. Find the volume of the cylindrical column standing on the area common to the parabolas $x = y^2$, $y = x^2$ as base and cut off by the surface $z = 12 + y - x^2$

(U.P.T.U 2001)

Ans. $\frac{569}{140}$

## OBJECTIVE PROBLEMS

Four alternative answers are given for each question, only one of them is correct. Tick mark the correct answer.

1. $\int_0^2 \int_4^6 (xy + e^x) dy\, dx$ is equal to

(i) $5 + e^2$ (ii) $2(9 + e^2)$

(iii) $2(7 + e^2)$ (iv) None of these

Ans. (ii)

2. $\int_0^1 \int_0^{\sqrt{y}} (x^2 + y^2) dy\, dx$ is equal to

(i) $\frac{7}{65}$ (ii) $\frac{44}{105}$

(iii) $\frac{64}{105}$ (iv) None of these

Ans. (ii)

3. If R s the region bounded by x =0, y =0, x + y =1, then $\iint_R (x^2 + y^2) dx\, dy$ is equal to

(i) $\frac{1}{3}$ (ii) $\frac{1}{5}$

(iii) $\frac{1}{6}$ (iv) $\frac{1}{12}$

Ans. (iii)

4. The area bounded by the parabola $y^2 = 4ax$, x-axis and the ordinates x =1, x =2 is given by

(i) $\frac{2}{3}\sqrt{a}(\sqrt{2} - 1)$

(ii) $\frac{4}{3}\sqrt{a}(2\sqrt{2} - 1)$

(iii) $\frac{2}{3}a(2\sqrt{2} + 1)$

(iv) None of these

Ans. (ii)

5. The area above the x - axis bounded by the curves $x^2 + y^2 = 2ax$ and $y^2 = ax$ is given by

(i) $a^2\left(\frac{\pi}{2}-\frac{2}{3}\right)$ (ii) $a^2\left(\frac{\pi}{4}-\frac{3}{2}\right)$

(iii) $a^2\left(\frac{\pi}{4}-\frac{2}{3}\right)$ (iv) None of these

Ans. (iii)

6. The area bounded by the curev xy = 4, y axis and the lines y =1 to y =4 is given by

(i) 2 log 2 (ii) 4 log 2

(iii) 8 log 2 (iv) None of these

Ans. (iii)

7. The volume of the area intercepted between the plane x + y + z = 1 and the Co-ordinate planes is

(i) $\frac{1}{2}$ (ii) $\frac{1}{3}$

(iii) $\frac{1}{6}$ (iv) None of these

Ans. (iii)

8. $\int_0^1\int_1^{x^2}\int_{2y}^{x+y} x\,dz\,dy\,dx$ is equal to

(i) $\frac{1}{6}$ (ii) $\frac{1}{10}$

(iii) $\frac{1}{30}$ (iv) $\frac{1}{15}$

Ans. (iii)

9. The volume of the tetrahedron bounded by the Co-ordinate planes and the plane x + y + z = 4 is equal to

(i) $\frac{32}{3}$ (ii) $\frac{16}{3}$ (iii) $\frac{4}{3}$ (iv) $\frac{128}{3}$

Ans. (i)

10. The volume of

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is

(i) $\frac{8}{3}\pi abc$ (ii) $\frac{4}{3}\pi abc$

(iii) $\frac{6}{5}\pi abc$ (iv) $\frac{4}{5}\pi abc$

Ans. (ii)

11. $\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz$ is equal to

(i) $(e - 1)^3$ (ii) $\frac{3}{2}(e - 1)$

(iii) $(e - 2)^2$ (iv) None of these

Ans. (i)

12. The volume of the tetrahedron bounded by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, a, b, c > 0 and the Co-ordinate planes is equal to

(i) $\frac{1}{3}\pi abc$ (ii) $\frac{2}{3}\pi abc$

(iii) $\frac{1}{6}\pi abc$ (iv) None of these

Ans. (iii)

13. The value of

$\int_0^1\int_{y^2}^1\int_0^{1-x} x\,dz\,dx\,dy$ is

(U.P.P.C.S. 1994)

(i) $\frac{4}{35}$ (ii) $\frac{3}{35}$ (iii) $\frac{8}{35}$ (iv) $\frac{6}{35}$

Ans. (i)

14. The value of $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dx\,dy$

(U.P.P.C.S. 1995)

(i) $\frac{\pi}{2}$ (ii) $\frac{\pi}{4}$

(iii) $\frac{\pi}{3}$ (iv) None of these

Ans. (ii)

15. The value of the integral $\int_0^1\int_0^1(x^2+y^2)dx\,dy$ is

(U.P.P.C.S. 1995)

(i) 1 (ii) 0

(iii) $\frac{1}{3}$ (iv) $\frac{2}{3}$

Ans. (iv)

16. The surface area of a sphere of radius r is

(R.A.S 1995)

(i) $4\pi r$ (ii) $4\pi r^2$

(iii) $6\pi r$ (iv) $8\pi r^2$

Ans. (ii)

# Chapter 11
# Beta and Gamma Functions

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**Introduction:** Beta and Gamma function are improper integrals which are commonly encountered in many science and engineering applications. These function are used in evaluating definite integrals. In this chapter we will study the beta and gamma function and apply them to some common problems.

**Definition:** The first and second Eulerian integrals which are also called Beta and Gamma functions respectively are defined as follows

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

and $\overline{|(n)} = \int_0^\infty e^{-x}x^{n-1}dx$

B(m, n) is read as Beta m, n and $\overline{|n}$ is read as Gamma n.

**Properties of Beta and Gamma Functions:**

**Property I:** B(m, n) = B (n, m) i.e. Beta functions is symmetrical with respect to m and n. ,

**Proof:** We know B(m, n)

$$= \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \int_0^1 (1-x)^{m-1}\left[1-(1-x)\right]^{n-1}dx$$

by the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

= B(n, m) Hence Proved

**Property II:** $\overline{|n} = (n-1)\overline{|(n-1)}$, for all values of n. (U.P.P.C.S. 1992)

**Proof:** We know that

$$\overline{|n} = \int_0^\infty e^{-x}x^{n-1}dx$$

$$= \left[x^{n-1}\left(-e^{-x}\right)\right]_0^\infty - \int_0^\infty (n-1)x^{n-2}\left(-e^{-x}\right)dx$$

integrating by part taking $x^{n-1}$ as first function

$$= 0 + (n-1)\int_0^\infty x^{n-2}e^{-x}dx$$

$$= (n-1)\overline{|(n-1)}$$ Hence Proved

Replacing n by (n + 1), we get $\overline{|(n+1)} = n\overline{|n}$

**Transformations of Gamma functions:**

(i) $\Gamma(n)=\int_0^1\left(\log\frac{1}{y}\right)^{n-1}dy$

**Proof:** We have

$$\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx$$

write $e^{-x}=y$ Then $-e^{-x}\,dx=dy$

$\Rightarrow -x\log e=\log y$

$\Rightarrow -x=\log y$

Now the equation (i) reduced to

$$\Gamma(n)=\int_1^\infty(-\log y)^{n-1}(-dy)$$

$$\Gamma(n)=\int_0^1\left(\log\frac{1}{y}\right)^{n-1}dy$$ Hence Proved

(ii) Prove that

$$\Gamma n=k^n\int_0^\infty e^{-ky}y^{n-1}dy$$

**Proof:** Since we know that by definition of Gamma function

$$\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx \qquad \text{(i)}$$

Suppose x =ky then dx = k dy

Now the equation (i) takes the form

$$\Gamma(n)=\int_0^\infty e^{-ky}(ky)^{n-1}kdy$$

$$=k^n\int_0^\infty e^{-ky}y^{n-1}dy$$

or $\Gamma(n)=k^n\int_0^\infty e^{-ky}y^{n-1}dy$ Hence Proved.

**Transformation of Beta functions:**

**(i) Prove that**

$$B(m,n)=\int_0^\infty\frac{y^{m-1}}{(1+y)^{m+n}}dy$$

(I.A.S. 1998, B.P.S.C. 2007, I.A.S. 2005)

**Proof:** Since we know that

$$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx \qquad \text{(i)}$$

write $x=\frac{1}{1+y}$ then $dx=-\frac{1}{(1+y)^2}dy$

with these values (i) becomes

$$B(m,n)=\int_{\infty}^{0}\left(\frac{1}{1+y}\right)^{m-1}\left(\frac{y}{1+y}\right)^{n-1}\left(\frac{-dy}{(1+y)^2}\right)$$

or $B(m,n)=\int_0^{\infty}\frac{y^{n-1}}{(1+y)^{m+n}}dy$ (ii)

Interchanging m and n, observing that B (m, n) = B (n, m) we have

$B(m,n)=\int_0^{\infty}\frac{y^{m-1}}{(1+y)^{m+n}}dy$ (iii)

Adding (ii) and (iii) we get

$$2B(m,n)=\int_0^{\infty}\frac{y^{m-1}+y^{n-1}}{(1+y)^{m+n}}dy$$

or $2B(m,n)=\int_0^{\infty}\frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$ (I.A.S. 2005, U.P.T.U. 2001)

**Relation between Beta and Gamma Functions:**

To Prove that

$$B(m,n)=\frac{\overline{|(m)}\,\overline{|(n)}}{\overline{|(m+n)}}$$

(I.A.S. 1990, I.A.S. 1991, B.P.S.C. 1993,97, 2005, U.P.T.U. 2001)

**Proof:** We know that

$\overline{|(n)}=\int_0^{\infty}e^{-x}x^{n-1}dx$ (i)

put x =ay then dx = ady

$\therefore\overline{|(n)}=\int_0^{\infty}e^{-ay}(ay)^{n-1}ady$

or $\frac{\overline{|(n)}}{a^n}=\int_0^{\infty}e^{-ay}y^{n-1}dy$

or $\frac{\overline{|(n)}}{a^n}=\int_0^{\infty}e^{-ax}x^{n-1}dx$ (ii)

or $\overline{|(n)}=a^n\int_0^{\infty}e^{-ax}x^{n-1}dx$

Putting a =y, we get

$\overline{|(n)}=y^n\int_0^{\infty}e^{-yx}x^{n-1}dx$

Multiplying both sides by $e^{-y}y^{m-1}dy$ and integrating,

$$\overline{|(n)}\int_0^{\infty}e^{-y}y^{m-1}dy=\int_0^{\infty}e^{-y}y^{m-1}y^n dy\int_0^{\infty}e^{-yx}x^{n-1}dx$$

$$\text{or } \Gamma n\,\Gamma m = \int_0^\infty \left(\int_0^\infty \left(e^{-y(1+x)} y^{m+n-1} dy\right) x^{n-1} dx\right.$$

$$\text{or } \Gamma n\,\Gamma m = \int_0^\infty \left\{\frac{\Gamma(m+n)}{(1+x)^{m+n}}\right\} x^{n-1} dx \text{ using (ii)}$$

$$\text{or } \frac{\Gamma m\,\Gamma n}{\Gamma m+n} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$= B(m, n)$$

Thus $B(m,n) = \dfrac{\Gamma m\,\Gamma n}{\Gamma m+n}$ Hence Proved.

**Some Important Deductions**

(i) To prove that $\Gamma n\,\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$

(I.A.S. 1990)

**Proof:** We know that

$$B(m,n) = \frac{\Gamma(m)\Gamma n}{\Gamma(m+n)}$$

Putting m+n =1 or m = 1- n, we get

$$\frac{\Gamma n\,\Gamma(1-n)}{\Gamma(1)} = B(n,1-n) \qquad \text{(i)}$$

we have $B(m,n) = \displaystyle\int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy$

$$\therefore B(n,1-n) = \int_0^\infty \frac{y^{n-1}}{1+y} dy = \frac{\pi}{\sin n\pi}, n < 1$$

$\therefore$ From (i) we have

$\Gamma(n)\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$ Hence Proved.

(ii) To prove that $\Gamma(1+n)\Gamma(1-n) = \dfrac{n\pi}{\sin n\pi}$

**Proof:** Since we know

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

Multiplying both sides by n, we get

$$n\Gamma(n)\Gamma(1-n) = \frac{n\pi}{\sin n\pi}$$

or $\Gamma(1+n)\,\Gamma(1-n)=\dfrac{n\pi}{\sin n\pi}$ Hence Proved

(iii) To prove that

$$\int_0^{\pi/2}\sin^m\theta\cos^n\theta d\theta=\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$$

**Proof:** we know that

$$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx$$

$$=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

putting x = $\sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$, we get

$$\int_0^1 x^{p-1}(1-x)^{q-1}dx=\int_0^{\pi/2}\left(\sin^2\theta\right)^{p-1}\left(1-\sin^2\theta\right)^{q-1}.2\sin\theta\cos\theta d\theta$$

$$=2\int_0^{\pi/2}\sin^{2p-1}\theta\cos^{2q-1}\theta d\theta$$

$$\therefore\int_0^{\pi/2}\sin^{2p-1}\theta\cos^{2q-1}\theta d\theta=\frac{1}{2}B(p.q)$$

or $\displaystyle\int_0^{\pi/2}\sin^{2p-1}\theta\cos^{2q-1}\theta d\theta=\frac{\Gamma p\,\Gamma q}{2\Gamma(p+q)}$

putting 2p-1 = m and 2 q - 1 = n

or $p=\dfrac{m+1}{2}$ and $q=\dfrac{n+1}{2}$

we get

$$\int_0^{\pi/2}\sin^m\theta\cos^n\theta d\theta=\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$$ Hence Proved.

## Legendre's Duplication Formula

To prove that

$$\Gamma(m)\Gamma\left(m+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2m-1}}.\Gamma(2m)$$

(I.A.S. 1993,1997, U.P.P.C.S. 1996, U.P.T.U. 2000)

**Proof:** Since we know that

$$\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta=\frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)} \qquad \text{(i)}$$

putting 2n-1 =0 or $n=\frac{1}{2}$, we get

$$\int_0^{\pi/2}\sin^{2m-1}\theta d\theta=\frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(m+\frac{1}{2}\right)}$$

$$=\frac{\Gamma(m)\sqrt{(\pi)}}{2\Gamma\left(m+\frac{1}{2}\right)} \qquad \text{(ii)}$$

putting n =m in (i), we get

$$\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2m-1}\theta d\theta=\frac{\{\Gamma(m)\}^2}{2\Gamma(2m)}$$

i.e. $\frac{1}{2^{2m}}\int_0^{\pi/2}(\sin 2\theta)^{2m-1}2d\theta=\frac{\{\Gamma(m)\}^2}{2\Gamma(2m)}$

putting $2\theta=\phi$ and $2d\theta=d\phi$, we get

$$\frac{1}{2^{2m}}\int_0^{\pi}\sin^{2m-1}\phi d\phi=\frac{\{\Gamma(m)\}^2}{2\Gamma(2m)}$$

or $\frac{2}{2^{2m}}\int_0^{\pi/2}\sin^{2m-1}\phi d\phi=\frac{\{\Gamma(m)\}^2}{2\Gamma(2m)}$

or $\int_0^{\pi/2}\sin^{2m-1}\phi d\phi=\frac{2^{2m-1}\{\Gamma(m)\}^2}{2\Gamma(2m)}$ (iii)

Equating two values of $\int_0^{\pi/2}\sin^{2m-1}\theta d\theta$ from (ii) and (iii) we get

$$\frac{2^{2m-1}\{\Gamma(m)\}^2}{2\Gamma(2m)}=\frac{\Gamma(m)\sqrt{\pi}}{2\Gamma\left(m+\frac{1}{2}\right)}$$

Hence $\Gamma(m)\Gamma\left(m+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2m-1}}.\Gamma(2m)$ Hence Proved

**Example 1:** Prove that

$$\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$$

**Solution:** Since we know that

$$\Gamma(n)=\int_0^\infty e^{-t}t^{n-1}dt$$

putting $t = x^2$ in Gamma function, we get

$$\Gamma(n)=2\int_0^\infty x^{2n-1}e^{-x^2}dx$$

Putting $n=\frac{1}{2}$ we have

$$\Gamma\left(\frac{1}{2}\right)=2\int_0^\infty e^{-x^2}dx$$

or $\sqrt{\pi}=2\int_0^\infty e^{-x^2}dx$

or $\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$ Hence Proved

**Example 2:** To prove that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

(I.A.S. 1990)

**Proof:** We know that

$$\Gamma n\,\Gamma 1-n=\frac{\pi}{\sin n\pi}$$

Putting $n=\frac{1}{2}$, we get

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{1}{2}\right)=\frac{\pi}{\sin\left(\frac{\pi}{2}\right)}$$

$$\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2=\pi$$

or $\Gamma\left(\frac{1}{2}\right)=\Gamma\pi$ Hence Proved.

**Example 3:** Show that

(U.P.P.C.S. 1995)

$$B(m, n) = B(m+1, n) + B(m, n+1)$$

**Solution:** R.H.S. = B(m+1, n) + B(m, n+1)

$$= \frac{\Gamma(m+1).\Gamma n}{\Gamma(m+1+n)} + \frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)}$$

$$= \frac{m\Gamma(m).\Gamma n}{(m+n)\Gamma(m+n)} + \frac{\Gamma m\,.n\Gamma(n)}{(m+n)\Gamma(m+n)}$$

$$= \frac{\Gamma m.\Gamma n}{\Gamma(m+n)}\left[\frac{m}{m+n} + \frac{n}{m+n}\right]$$

$$= \frac{\Gamma m\,,\Gamma n}{\Gamma(m+n)}$$

= B(m, n)

= L.H.S. Hence Proved

**Example 4:** Show that

$$\int_0^{\infty} \frac{x^c}{c^x} dx = \frac{\Gamma(c+1)}{(\log c)} c+1$$

**Solution:**

$$\text{Let } I = \int_0^{\infty} \frac{x^c}{c^x} dx$$

put $c^x = e^t$

$\Rightarrow x \log c = t$

$$\Rightarrow \quad x = \frac{t}{\log c}$$

$$\Rightarrow dx = \frac{dt}{\log c}$$

$$\text{so, } \quad I = \int_0^{\infty} \left(\frac{t}{\log c}\right)^c \frac{1}{e^t} \frac{dt}{\log c}$$

$$= \frac{1}{(\log c)} c+1 \int_0^{\infty} e^{-t} t^c dt$$

$$= \frac{1}{(\log c)} c+1 \int_0^{\infty} e^{-t} t^{c+1-1} dt$$

$$= \frac{\Gamma(c+1)}{(\log c)^{c+1}}$$ Hence Proved.

**Example 5:**

Prove that

$$\Gamma\left(\frac{3}{2}-x\right).\Gamma\left(\frac{3}{2}+x\right)=\left(\frac{1}{4}-x^2\right)\pi\sec\pi x$$

**Solution:** L.H.S. $\Gamma\left(\frac{3}{2}-x\right).\Gamma\left(\frac{3}{2}+x\right)=\left(\frac{1}{2}-x\right)\Gamma\left(\frac{1}{2}-x\right).\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}+x\right)$

$\because \Gamma(n+1)=n\Gamma n$

$$=\left(\frac{1}{4}-x^2\right)\Gamma\left(\frac{1}{2}-x\right).\Gamma\left(\frac{1}{2}+x\right)$$

$$=\left(\frac{1}{4}-x^2\right)\Gamma\left(\frac{1}{2}-x\right)\Gamma\left\{1-\left(\frac{1}{2}-x\right)\right\}$$

$$=\left(\frac{1}{4}-x^2\right)\frac{\pi}{\sin\left(\frac{1}{2}-x\right)\pi}$$

$$\because \Gamma(n)\,\Gamma(1-n)=\frac{\pi}{\sin n\pi}$$

$$=\left(\frac{1}{4}-x^2\right).\frac{\pi}{\sin\left(\frac{\pi}{2}-\pi x\right)}$$

$=\left(\frac{1}{4}-x^2\right)\pi\sec(\pi x)$ Hence Proved

= R.H.S.

**Example 6:** Show that

$$\int_0^{\pi/2}\tan^n\theta\,d\theta=\frac{\pi}{2}\sec\left(\frac{n\pi}{2}\right)$$

**Solution:** We have

$$\int_0^{\pi/2}\tan^n\theta d\theta=\int_0^{\pi/2}\sin^n\theta(\cos\theta)^{-n}\,d\theta$$

$$=\frac{\Gamma\left\{\frac{1}{2}(n+1)\right\}.\Gamma\left\{\frac{1}{2}(-n+1)\right\}}{2\Gamma\left\{\frac{1}{2}(n-n+2)\right\}}$$

$$=\frac{\Gamma\left(\frac{n+1}{2}\right).\Gamma\left(1-\frac{n+1}{2}\right)}{2\Gamma 1}$$

$$\because \Gamma n\,\Gamma(1-n)=\frac{\pi}{\sin n\pi}$$

$$=\frac{1}{2}\frac{\pi}{\sin\frac{1}{2}(n+1)\pi}$$

$$=\frac{\pi}{2}\operatorname{cosce}\left\{\frac{1}{2}(n+1)\pi\right\}$$

$$=\frac{\pi}{2}\operatorname{cosce}\left(\frac{\pi}{2}+\frac{n\pi}{2}\right)$$

$$=\frac{\pi}{2}\sec\left(\frac{n\pi}{2}\right)$$ Hence Proved.

**Example 7:** Show that

$$B(n,n)=\frac{\sqrt{\pi}\,\Gamma(n)}{2^{2n-1}\Gamma\left(n+\frac{1}{2}\right)}$$

(U.P.P.C.S. 1990)

**Proof:** We know that

$$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

put $x=\sin^2\theta \Rightarrow dx=2\sin\theta\cos\theta\,d\theta$

$$\therefore B(m,n)=\int_0^{\pi/2}(\sin\theta)^{2m-2}(\cos\theta)^{2n-2}.2\sin\theta\cos\theta\,d\theta$$

$$=2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta \qquad \text{(i)}$$

$$=\frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}$$

Hence $B(m,n)=\dfrac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}$

From (i) we have

$$B(n,n) = 2\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2n-1}\theta\, d\theta$$

$$= \frac{2}{2^{2n-1}} \int_0^{\pi/2} (2\sin\theta\cos\theta)^{2n-1}\, d\theta$$

$$= \frac{2}{2^{2n-1}} \int_0^{\pi/2} \sin^{2n-1} 2\theta\, d\theta$$

$$= \frac{2}{2^{2n-1}} \cdot \frac{1}{2} \int_0^{\pi} \sin^{2n-1}\phi\, d\phi$$

$$= \frac{2}{2^{2n-1}} \cdot \frac{1}{2} . 2\int_0^{\pi/2} \sin^{2n-1}\phi\, d\phi$$

$$= \frac{2}{2^{2n-1}} \frac{\Gamma(n)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(n+\frac{1}{2}\right)}$$

or $B(n,n) = \dfrac{\sqrt{\pi}\,\Gamma(n)}{2^{2n-1}\Gamma\left(n+\frac{1}{2}\right)}$ Hence Proved.

**Example 8:** Find the value of $\Gamma\frac{1}{2}$

**Solution:** we know that

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{\Gamma\frac{p+1}{2}\Gamma\frac{q+1}{2}}{2\Gamma\frac{p+q+2}{2}}$$

putting p = q =0 we have

$$\int_0^{\pi/2} d\theta = \frac{\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{2\Gamma 1}$$

$$\Rightarrow [\theta]_0^{\pi/2} = \frac{1}{2}\left(\Gamma\frac{1}{2}\right)^2$$

$$\Rightarrow \quad \frac{\pi}{2} = \frac{1}{2}\left(\Gamma\frac{1}{2}\right)^2$$

$$\Rightarrow \left(\Gamma\frac{1}{2}\right)^2 = \pi$$

$$\Rightarrow \overline{\left|\frac{1}{2}\right.} = \sqrt{\pi} \qquad \text{Answer.}$$

**Example 9:** To show that

(i) $\int_0^\infty e^{-ax}x^{n-1}\cos bx\,dx = \frac{\overline{|(n)}\cos n\theta}{\left(a^2+b^2\right)^{\frac{n}{2}}}$

(U.P.T.U. C.C.O. 2004)

(ii) $\int_0^\infty e^{-ax}\sin bx\,dx = \frac{\overline{|(n)}\sin n\theta}{\left(a^2+b^2\right)^{\frac{n}{2}}}$

(U.P.T.U. 2003)

**Solution:** we know that

$\int_0^\infty e^{-ax}x^{n-1}dx = \frac{\overline{|(n)}}{a^n}$, where a, n are positive put ax =z so that $dx = \frac{dz}{a}$

$$\int_0^\infty e^{-ax}x^{n-1}dx = \int_0^\infty e^{-z}\left(\frac{z}{a}\right)^{n-1}\frac{dz}{a}$$

$$= \frac{1}{a^n}\int_0^\infty e^{-z}z^{n-1}dz = \frac{\overline{|(n)}}{a^n}$$

Replacing a by a + ib, we have

$$\int_0^\infty e^{-(a+ib)x}\,x^{n-1}dx = \frac{\overline{|(n)}}{(a+ib)^n}$$

Now $e^{-(a+ib)x} = e^{-ax}.\,e^{-ibx}$

$= e^{-ax}(\cos bx - i\sin bx)$

Putting $a = r\cos\theta$ and $b = r\sin\theta$ so that

$r^2 = a^2 + b^2$ and $\theta = \tan^{-1}\frac{b}{a}$

$(a + ib)^n = (r\cos\theta + ir\sin\theta)^n$

$= r^n(\cos\theta + i\sin\theta)^n$

$= r^n(\cos n\theta + i\sin n\theta)$

(De Moivre's theorem)

∴ From (i) we get

$$\int_0^\infty e^{-ax}(\cos bx - i\sin bx)x^{n-1}dx = \frac{\overline{|(n)}}{r^n(\cos n\theta + i\sin n\theta)}$$

$$=\frac{\Gamma(n)}{r^n}(\cos n\theta+i\sin n\theta)^{-1}$$

$$=\frac{\Gamma(n)}{r^n}(\cos n\theta-i\sin n\theta)$$

Now equating real and imaginary parts on the two sides, we get

$$\int_0^\infty e^{-ax}x^{n-1}\cos bx\,dx=\frac{\Gamma(n)}{r^n}\cos n\theta$$

and $\int_0^\infty e^{-ax}x^{n-1}\sin bx\,dx=\frac{\Gamma(n)}{r^n}\sin n\theta$

where $r^2 = a^2 + b^2$ and $\theta = \tan^{-1}\frac{b}{a}$

**Example 10:** Prove that

$$\iint_D x^{l-1}y^{m-1}dx\,dy=\frac{\Gamma l\,\Gamma m}{\Gamma(l+m+1)}h^{l+m}$$

where D is the domain $x \geq 0$, $y \geq 0$ and $x + y \leq h$

(U.P.T.U. 2005)

**Solution:** Putting $x = Xh$ and $y = Yh$, we get

$dx\,dy = h^2\,dX\,dY$

Therefore,

$$\iint_D x^{l-1}y^{m-1}dx\,dy=\iint_{D'}(Xh)^{l-1}(Yh)^{m-1}h^2dX\,dY$$

where D' is the domain $X \geq 0$, $Y \geq 0$, $X + Y \leq 1$

$$\iint_D x^{l-1}y^{m-1}dx\,dy=h^{l+m}\int_0^1\int_0^{1-x}X^{l-1}Y^{m-1}dX\,dY$$

$$=h^{l+m}\int_0^1 X^{l-1}\left[\frac{Y^m}{m}\right]_0^{1-x}dX$$

$$=\frac{h^{l+m}}{m}\int_0^1 X^{l-1}(1-X)^m\,dX$$

$$=\frac{h^{l+m}}{m}B(l.m+1)$$

$$=\frac{h^{l+m}}{m}\frac{\Gamma(l)\Gamma(m+1)}{\Gamma(l+m+1)}$$

$$=\frac{h^{l+m}\Gamma(l)\,m\,\Gamma m}{m\,\Gamma(l+m+1)}$$

$$=\frac{h^{l+m}\Gamma(l)m\Gamma m}{\Gamma(l+m+1)}$$ Hence Proved.

**Example 11:** Prove that $\int_0^{\pi/2}\frac{d\theta}{\sqrt{\sin\theta}}\times\int_0^{\pi/2}\sqrt{\sin\theta}\,d\theta=\pi$

**Solution:**

$$\int_0^{\pi/2}\frac{d\theta}{\sqrt{\sin\theta}}=\int_0^{\pi/2}(\sin\theta)^{-1/2}\cos^0\theta\,d\theta$$

$$=\frac{\Gamma\left\{\frac{1}{2}\left(-\frac{1}{2}+1\right)\right\}\Gamma\left\{\frac{1}{2}(0+1)\right\}}{2\Gamma\left\{\frac{1}{2}\left(-\frac{1}{2}+0+2\right)\right\}}$$

$$=\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{3}{4}\right)} \qquad \text{(i)}$$

and $\int_0^{\pi/2}\sqrt{(\sin\theta)}\,d\theta=\int_0^{\pi/2}\sin^{1/2}\theta\cos^0\theta\,d\theta$

$$=\frac{\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+1\right)\right\}\Gamma\left\{\frac{1}{2}(0+1)\right\}}{2\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+0+2\right)\right\}}=\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{5}{4}\right)}$$

$\therefore$ From (i) and (ii) we have

$$\int_0^{\pi/2}\frac{d\theta}{\sqrt{(\sin\theta)}}\times\int_0^{\pi/2}\sqrt{(\sin\theta)}\,d\theta=\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{3}{4}\right)}\times\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{5}{4}\right)}$$

$$=\frac{\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}.\sqrt{\pi}}{4.\frac{1}{4}.\Gamma\left(\frac{1}{4}\right)}=\pi$$ Hence Proved.

## Dirichlet's Theorem for Three Variables:

If *l*, m, n are all positive, then the triple integral

$$\iiint_V x^{l-1}y^{m-1}z^{n-1}dx\,dy\,dz=\frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}$$

where V is the region $x \geq 0,\ y \geq 0\ z \geq 0$ and $x + y + z \leq 1$

(U.P.T.U. 2005)

**Poof:** Putting $y + z \leq 1 - x = h$. Then $z \leq h - y$

$$\iiint_V x^{l-1}y^{m-1}z^{n-1}dxdydz=\int_0^1 x^{l-1}dx\int_0^{1-x}y^{m-1}dy\int_0^{1-x-y}z^{n-1}dz$$

$$=\int_0^1 x^{l-1}dx\left[\int_0^h\int_0^{h-y}y^{m-1}z^{n-1}dydz\right]$$

(put x = h)

$$=\int_0^1 x^{l-1}dx\left[\frac{\Gamma m\,\Gamma n}{\Gamma(m+n+1)}h^{m+n}\right]$$

$$=\frac{\Gamma m\,\Gamma n}{\Gamma(m+n+1)}\int_0^1 x^{l-1}(1-x)^{m+n}dx$$

$$=\frac{\Gamma m\,\Gamma n}{\Gamma m+n+1}\beta(l,m+n+1)$$

$$=\frac{\Gamma m\,\Gamma n}{\Gamma(m+n+1)}\frac{\Gamma l\,\Gamma(m+n+1)}{\Gamma(l+m+n+1)}$$

$$\Rightarrow\iiint_V x^{l-1}y^{m-1}z^{n-1}dx\,dy\,dz=\frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}$$

Hence Proved.

**Note 1:** $\iiint_V x^{l-1}y^{m-1}z^{n-1}dx\,dy\,dz=\frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}h^{l+m+n}$

Where V is the domain, $x \geq 0,\ y \geq 0\ z \geq 0$ and $x + y + z \leq h$

**Note 2:** Dirichlet's theorem for n variable, the theorem states that

$$\iiint\ldots\ldots\ldots\int x_1^{l_1-1}x_2^{l_2-1}\ldots\ldots\ldots x_n^{l_n-1}dx_1dx_2\ldots\ldots\ldots\ldots dx_n=\frac{\Gamma l_1\,\Gamma l_2\,\Gamma l_3\ldots\ldots\Gamma l_n}{\Gamma(1+l_1+l_2+\ldots.+l_n)}h^{l_1+l_2+\ldots+l_n}$$

**Example:** State the Dirichlet's theorem for three variables. Hence evaluate the integral $\iiint x^{l-1}y^{m-1}z^{n-1}dx\,dy\,dz$ where x, y, z are all positive with conditions $\left(\frac{x}{a}\right)^p+\left(\frac{y}{b}\right)^q+\left(\frac{z}{c}\right)^r\leq 1$

(U.P.T.U. 2005)

**Solution:** The required integral $\iiint x^{l-1}y^{m-1}z^{n-1}dxdy\,dz$ where the integral is extended to all positive values of the variables x, y and z subject to the condition $\left(\frac{x}{a}\right)^p+\left(\frac{y}{b}\right)^q+\left(\frac{z}{c}\right)^r \le 1$ Let us put

$\left(\frac{x}{a}\right)^p$ = u i.e. x $=au^{1/p}$ so that $dx=\left(\frac{a}{p}\right)u^{\frac{1}{p}-1}du$

$\left(\frac{y}{b}\right)^q$ = v i.e. y = $bv^{1/q}$ so that $dy=\left(\frac{b}{q}\right)v^{\frac{1}{q}-1}dv$

and $\left(\frac{z}{c}\right)^r$ = w i.e. z = $cw^{1/r}$ so that $dx=\left(\frac{c}{r}\right)w^{\frac{1}{r}-1}dw$

Required integral

$$=\iiint\left(au^{1/p}\right)^{l-1}\left(bv^{1/q}\right)^{m-1}\left(cw^{1/r}\right)^{n-1}\left(\frac{a}{p}\right)u^{\frac{1}{p}-1}du\left(\frac{b}{q}\right)v^{\frac{1}{q}-1}dv\left(\frac{c}{r}\right)w^{\frac{1}{r}-1}dw$$

$$=\iiint\left(a^{l-1}u^{\frac{l-1}{p}}\right)\left(b^{m-1}v^{\frac{m-1}{q}}\right)\left(c^{n-1}w^{\frac{n-1}{r}}\right).\frac{a}{p}u^{\frac{1}{p}-1}.\frac{b}{q}v^{\frac{1}{q}-1}.\frac{c}{r}w^{\frac{1}{r}-1}du\,dv\,dw$$

where u + v + w $\le 1$

$$=\frac{a^l b^m c^n}{pqr}\iiint u^{\frac{l}{p}-1}u^{\frac{m}{q}-1}w^{\frac{n}{r}-1}du\,dv\,dw$$

$$=\frac{a^l b^m c^n}{pqr}\frac{\Gamma\frac{l}{p}\Gamma\frac{m}{q}\Gamma\frac{n}{r}}{\Gamma\left(\frac{l}{p}+\frac{m}{q}+\frac{n}{r}+1\right)}$$ by Dirichlet's integral

Answer.

**Example :** Find the mass of an octant of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, the density at any point being p = kxyz

(U.P.T.U. 2002, 2006)

**Solution:** We know that

Mass $=\iiint \rho\,dv$

$=\iiint (kxyz)dx\,dy\,dz$

$=k\iiint (xdx)(ydy)(zdz)$ (i)

Putting $\frac{x^2}{a^2} = u, \frac{y^2}{b^2} = v, \frac{z^2}{c^2} = w$ and $u + v + w = 1$

so that $\frac{2xdx}{a^2} = du, \frac{2ydy}{b^2} = dv, \frac{2zdz}{c^2} = dw$

$$\text{Mass} = k\iiint\left(\frac{a^2du}{2}\right)\left(\frac{b^2dv}{2}\right)\left(\frac{c^2dw}{2}\right)$$

$$= \frac{k\,a^2b^2c^2}{8}\iiint du\,dv\,dw \text{ where } u + v + w \leq 1$$

$$= \frac{ka^2b^2c^2}{8}\iiint u^{1-1}v^{1-1}w^{1-1}du\,dv\,dw$$

$$= \frac{ka^2b^2c^2}{8}\frac{\Gamma 1\,\Gamma 1\,\Gamma 1}{\Gamma(3+1)} = \frac{ka^2b^2c^2}{8\times 6}$$

$$= \frac{ka^2b^2c^2}{48} \qquad \text{Answer.}$$

**Example:** The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ meets the axes in A, B and C. Apply Dirichlet's integral to find the volume of the tetrahedran OABC. Also find its mass if the density at any point is kxyz.

(U.P.T.U. 2004)

**Solution:** The volume of the tetrahedron OABC is given by $V = \iiint_D dxdydz$ for all is positive values of x, y and z subjected to the condition $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

putting $\frac{x}{a} = u \Rightarrow dx = adu$, $\frac{y}{b} = v \Rightarrow dy = bdv$ and $\frac{z}{c} = w \Rightarrow dz = cdw$, we get

$$V = \iiint_{D'} abc\,du\,dv\,dw$$

where $u \geq 0, v \geq 0, w \geq 0$ subjected to the condition $u + v + w \leq 1$

$$V = abc\iiint_{D'} u^{1-1}v^{1-1}w^{1-1}du\,dv\,dw$$

Applying Dirichlet's integral, we get

$$V = abc\frac{\Gamma 1\,\Gamma 1\,\Gamma 1}{\Gamma(1+1+1+1)} = \frac{abc}{\Gamma 4}$$

$$= \frac{abc}{6}$$

Now mass $= \iiint\limits_{D} k\,xyz\,dx\,dy\,dz$

$= k\iiint\limits_{D'} au\,bv\,cw\,(abc)\,du\,dv\,dw$

$= ka^2b^2c^2\iiint\limits_{D'} uvw\,du\,dv\,dw$

$= \dfrac{ka^2b^2c^2\,\Gamma 2\,\Gamma 2\,\Gamma 2}{\Gamma(2+2+2+1)} = \dfrac{ka^2b^2c^2}{\Gamma 7}$

$= \dfrac{ka^2b^2c^2}{720}$ Answer.

**Example:** Evaluate $I = \iiint\limits_{V} x^{\alpha-1}y^{\beta-1}z^{\gamma-1}dx\,dy\,dz$

where V is the region in the first octant bounded by sphere $x^2 + y^2 + z^2 = 1$ and the Co-ordinate planes.

[U.P.T.U. (C.O.) 2003]

**Solution:**

Let $x^2 = u \Rightarrow x = \sqrt{u}$ therefore $dx = \dfrac{1}{2\sqrt{u}}du$, $y^2 = v \Rightarrow y = \sqrt{v}$, therefore $dy = \dfrac{1}{2\sqrt{v}}dv$, $z^2 = w \Rightarrow z = \sqrt{w}$ therefore $dz = \dfrac{1}{2\sqrt{w}}dw$

Then $u + v + w = 1$ Also, $u \geq 0, v \geq 0, w \geq 0$

$I = \iiint\limits_{V} \left(\sqrt{u}\right)^{\alpha-1}\left(\sqrt{v}\right)^{\beta-1}\left(\sqrt{w}\right)^{\gamma-1}\dfrac{du}{2\sqrt{u}}\cdot\dfrac{dv}{2\sqrt{v}}\cdot\dfrac{dw}{2\sqrt{w}}$

$= \dfrac{1}{8}\iiint u^{\frac{\alpha}{2}-1}v^{\frac{\beta}{2}-1}w^{\frac{\gamma}{2}-1}du\,dv\,dw$

$= \dfrac{1}{8}\dfrac{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}+1\right)}$ Answer.

## LIOUVILLE'S EXTENSION OF DIRICHLET THEOREM:

If the variables x, y, z are all positive such the $h_1 < (x + y + z) < h_2$ then

$$\iiint f(x+y+z)x^{l-1}y^{m-1}z^{n-1}dx\,dy\,dz = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n)}\int_{h_1}^{h_2} f(u)u^{l+m+n-1}du$$

**Proof:** Let $I = \iiint x^{l-1}y^{m-1}z^{n-1}dx\,dy\,dz$ under the condition $x + y + z \leq u$ then

$$I=\left(\frac{\Gamma(l)+\Gamma(m)+\Gamma(n)}{\Gamma(l+m+n+1)}\right)u^{l+m+n} \qquad \text{(i)}$$

(by Dirichlet's theorem)

if x + y + z ≤ u + δu, then

$$I=(u+\delta u)^{l+m+n}\frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)} \qquad \text{(ii)}$$

Now if u< x + y + z < (u + δu) then

$$\iiint x^{l-1}y^{m-1}z^{n-1}dx\,dy\,dz=\frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}\left[(u+\delta u)^{l+m+n}-u^{l+m+n}\right]$$

$$=\frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}u^{l+m+n}\left[1+\left(\frac{\delta u}{u}\right)^{l+m+n}-1\right]$$

$$=\frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(1+m+n+1)}u^{1+m+n}\left[1+(l+m+n)\frac{\delta u}{u}+.........-1\right]$$

$$=\frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}u^{l+m+n}(l+m+n)\frac{\delta u}{u}$$

$$=\frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}u^{l+m+n}\delta u$$

Now consider $\iiint f(x+y+z)x^{l-1}y^{m-1}z^{n-1}dxdydz$. Under the condition $h_1 \le (x + y + z) \le h_2$. When x + y + z lies between u and u + δu, the value of f(x + y + z) can only differ from f(u) by a small quantity of the same order as δu. Hence

$$\iiint f(x+y+z)x^{l-1}y^{m-1}z^{n-1}dx\,dy\,dz=\frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}\int f(u)u^{l+m+n-1}\delta u$$

Where x + y + z lies between u and u + δu

Therefore

$$\iiint f(x+y+z)x^{l-1}y^{m-1}z^{n-1}dx\,dy\,dz=\frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}\int_{h_1}^{h_2} f(u)u^{l+m+n-1}du$$

**Example:** Evaluate

$\iiint \log(x+y+z)dx\,dy\,dz$, the integral extending over all positive and zero values of x, y , z subjected to x + y + z < 1

**Solution:** $0 \le x + y + z < 1$

$$\therefore \iiint \log(x+y+z)dx\,dy\,dz=\iiint x^{1-1}y^{1-1}z^{1-1}\log(x+y+z)dx\,dy\,dz$$

$$= \frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(3)} \int_0^1 t^{1+1+1-1} \log t\, dt$$

by Liouvill's extension of Dirichlet's theorem

$$= \frac{1}{2}\int_0^1 t^2 \log t\, dt$$

$$= \frac{1}{2}\left[\left(\frac{t^3}{3}\log t\right)_0^1 - \int_0^1 \frac{t^3}{2}.\frac{1}{t}dt\right]$$

$$= \frac{1}{2}\left[-\frac{1}{3}\left(\frac{t^3}{3}\right)_0^1\right] = -\frac{1}{18} \qquad \text{Answer.}$$

**Example:** Evaluate

$$\iiint \ldots\ldots\ldots\ldots \int_n \frac{dx_1\, dx_2 \ldots\ldots\ldots\ldots dx_n}{\sqrt{1 - x_1^2 - x_2^2 \ldots\ldots\ldots x_n^2}}$$

integral being extended to all positive values of the variables for which the expression is real.

(U.P.T.U. 2001)

**Solution:** The expression will be real, if

$1 - x_1^2 - x_2^2 - \ldots\ldots\ldots\ldots\ldots\ldots - x_n^2 > 0$

or $x_1^2 + x_2^2 + \ldots\ldots\ldots\ldots\ldots\ldots + x_n^2 < 1$

Hence the given integral is extended for all positive value of the variables $x_1$, $x_2 \ldots\ldots\ldots\ldots x_n$ such that

$0 < x_1^2 + x_2^2 + \ldots\ldots\ldots\ldots\ldots\ldots x_n^2 < 1$

Let us put $x_1^2 = u_1$ i.e. $x_1 = \sqrt{u_1}$ so that, $dx_1 = \frac{1}{2\sqrt{u_1}} du_1$ etc.

Then the condition becomes, $0 < u_1 + u_2 + \ldots\ldots\ldots\ldots + u_n < 1$

$$\therefore \text{ required integral } = \frac{1}{2^n}\iiint \ldots\ldots\ldots \int_n \frac{u_1^{-1/2} u_2^{-1/2} \ldots\ldots\ldots u_n^{-1/2}}{\sqrt{1 - u_1 - u_2 \ldots\ldots\ldots u_n}} du_1 du_2 \ldots\ldots du_n$$

$$= \frac{1}{2^n}\iiint \ldots\ldots\ldots \int_n \frac{u_1^{\frac{1}{2}-1} u_2^{\frac{1}{2}-1} \ldots\ldots\ldots u_n^{\frac{1}{2}-1}}{\sqrt{1 - u_1 - u_2 \ldots\ldots\ldots - u_n}} du_1 du_2 \ldots\ldots du_n$$

$$= \frac{1}{2^n} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\ldots\ldots\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+\ldots\ldots\ldots+\frac{1}{2}\right)} \int_0^1 \frac{1}{\sqrt{1-u}}.\, u^{\left(\frac{1}{2}+\frac{1}{2}+\ldots\ldots+\frac{1}{2}\right)-1} du$$

By Liouville's Extension

$$=\frac{1}{2^n}\frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^n}{\Gamma\left(\frac{n}{2}\right)}\int_0^1\frac{1}{\sqrt{1-u}}\,u^{\frac{n}{2}-1}du$$

$$=\frac{1}{2^n}\frac{\left(\sqrt{\pi}\right)^n}{\Gamma(n/2)}\int_0^{\pi/2}\frac{1}{\sqrt{1-\sin^2\theta}}\left(\sin^2\theta\right)^{\frac{n}{2}-1}.2\sin\theta\cos d\theta$$

putting u = sin²θ

$$=\frac{1}{2^{n-1}}\frac{\left(\sqrt{\pi}\right)^n}{\Gamma(n/2)}\int_0^{\pi/2}\sin^{n-1}\theta d\theta$$

$$=\frac{1}{2^{n-1}}.\frac{(\pi)^{n/2}}{\Gamma(n/2)}.\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{n+1}{2}\right)}$$

$$=\frac{1}{2^n}.\frac{(\pi)^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}\quad\text{Answer.}$$

**EXERCISE**

1. Evaluate (i) $\Gamma\left(-\frac{1}{2}\right)$ (ii) $\Gamma\left(-\frac{3}{2}\right)$

Ans. (i) $-2\sqrt{\pi}$ (ii) $\frac{4}{3}\sqrt{\pi}$

2. $\int_0^1 x^3(1-x)^4\,dx$

Ans. $\frac{1}{280}$

3. Prove that $\int_0^\infty e^{-ay}\,y^{n-1}dy=\frac{\Gamma(n)}{a^n}$

4. Show that $\int_0^{\pi/2}\sin^3 x\cos^{5/2}x\,dx=\frac{8}{77}$

5. Evaluate $\int_0^1\frac{x^2dx}{\sqrt{1-x^3}}$

Ans. $\frac{2}{3}$

6. Express $\int_0^1 x^m\left(1-x^n\right)^p dx$ in terms of the beta function, and hence evaluate $\int_0^1 x^5\left(1-x^3\right)^{10} dx$

Ans. (i) $\frac{1}{n}\beta\left(\frac{m+1}{n},p+1\right)$ (ii) $\frac{1}{396}$

7. Evaluate $\int_0^\infty \frac{x^4\left(1+x^5\right)}{(1+x)^{15}}dx$

Ans. $\frac{1}{5005}$

8. Prove that $\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}.\Gamma(2n+1)}{2^{2n}\Gamma(n+1)}$

9. Evaluate $\int_0^1 \log\Gamma x\, dx$

Ans. $\frac{1}{2}\log 2\pi$

10. Evaluate $\int_0^\infty x^8\frac{\left(1-x^6\right)}{(1+x)^{24}}dx$

Ans. 0

11. Evaluate $\int_0^\infty \frac{x^3\left(1+x^4\right)}{(1+x)^{10}}dx$

Ans. $\frac{1}{63}$

12. Evaluate $\int_{-\infty}^\infty \cos\frac{\pi}{2}x^2 dx$

Ans. 1

13. Evaluate

(i) $\int_0^\infty x^{1/4}e^{-\sqrt{x}}dx$

(ii) $\int_0^\infty x^2 e^{-a^2x^2}dx$

Ans. $\frac{\sqrt{\pi}}{4a^3}$

14. Evaluate $\int_0^1 x^{m-1}\left(\log\frac{1}{x}\right)^{n-1} dx, m > 0, n > 0$

Ans. $\frac{\left|\overline{n}\right.}{m^n}$

15. Find the volume of the solid surrounded by the surface

$$\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}+\left(\frac{z}{c}\right)^{2/3}=1$$

(U.P.T.U. 2008)
Ans. $4\pi abc/35$

16. Evaluate $\iiint dx\,dy\,dz$ where $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$

Ans. $\frac{\pi abc}{6}$

17. show that $\iiint \frac{dxdydz}{(x+y+z+1)^3}=\frac{1}{2}\log 2-\frac{5}{16}$ the integral being taken thought the volume bounded by the planes $x = 0, y = 0, z = 0, x + y + z + 1$

**OBJECTIVE PROBLEMS**

Four alternative answers are given for each question, only one of them is correct. Tick mark the correct answer.

1. $\int_0^\infty e^{-x}x^3$ is equal to

(i) 2 (ii) 4
(ii) 6 (iv) None of these

Ans. (iii)

2. When n is a positive integer then $\int_0^1\left(\log\frac{1}{x}\right)^{n-1} dx$ is equal to

(i) $\left|\overline{n-1}\right.$ (ii) $\left|\overline{n}\right.$
(ii) $\left|\overline{(n+1)}\right.$ (iv) $\frac{1}{2}\left|\overline{(n+1)}\right.$

Ans. (ii)

3.When $-\frac{1}{2} < n < \frac{1}{2}$, then the value of $\left|\overline{\left(\frac{3}{2}-n\right)}\right.\left|\overline{\left(\frac{3}{2}+n\right)}\right.$ is

(i) $\left(\frac{1}{4}-n^2\right)\pi \sec n\pi$ (ii) $\left(\frac{1}{4}-n^2\right)\pi\,\text{cosec}\, n\pi$

(ii) $\left(\frac{1}{4}-n^2\right)\pi \sin n\pi$ (iv) None of these

Ans. (i)

4. The value of $\left|\overline{(.1)}\right.\left|\overline{(.2)}\right.\left|\overline{(.3)}\right.\ldots\ldots\left|\overline{(.9)}\right.$ is

(i) $\frac{2\pi}{\sqrt{10}}$ (ii) $\frac{(2\pi)^{10/2}}{\sqrt{10}}$

(ii) $\frac{(2\pi)^{9/2}}{\sqrt{10}}$ (iv) None of these

Ans. (iii)

5. $\int_0^1 \frac{dx}{\sqrt{1-x}}$ is equal to

(i) B (1, 1) (ii) $B\left(\frac{1}{2},\frac{1}{2}\right)$

(iii) $B\left(0,\frac{3}{2}\right)$ (iv) $B\left(1,\frac{1}{2}\right)$

Ans. (iv)

6. B(2, 3) is equal to

(i) $\frac{1}{10}$ (ii) $\frac{1}{12}$

(ii) $\frac{1}{24}$ (iv) None of these

Ans. (ii)

7. $\int_0^{\pi/2} \sin^9 x\,dx$ is equal to

(i) $\frac{64}{65}$ (ii) $\frac{64}{35}$

(ii) $\frac{64}{315}$ (iv) $\frac{128}{315}$

Ans. (iv)

8. $\int_0^{\pi} \sin^3 x\,(1-\cos x)^2\,dx$ is equal to

(i) $\frac{2}{3}$ (ii) $\frac{2}{5}$

(ii) $\frac{4}{5}$ (iv) $\frac{8}{5}$

Ans. (iv)

9. The value of

$\int_0^a \sqrt{\frac{a-x}{x}}\,dx$ is

(M.P.P.C.S. 1995)

(i) $\frac{a}{2}$ (ii) $\frac{a}{4}$

(ii) $\frac{\pi a}{2}$ (iv) $\frac{\pi a}{4}$

Ans. (iii)
(M.P.PC.S. 1995)

10. $\int_0^{\pi/2} \cos^5\theta \sin^3\theta \, d\theta$ is equal to

(i) $\frac{1}{32}$ (ii) $\frac{\pi}{24}$

(ii) $\frac{1}{16}$ (iv) $\frac{1}{24}$

Ans. (iv)

11. $\int_0^{\infty} e^{-a^2x^2} dx$ is equal to

(i) $\sqrt{\pi}$ (ii) $\frac{\sqrt{\pi}}{2a}$

(ii) $-\frac{\sqrt{\pi}}{4a}$ (iv) $\frac{\pi}{2a}$

Ans. (ii)
(U.P.P.C.S. 1994)

12. Match the list I with list II

| List I | List II |
|---|---|
| (a) $\int_0^{\infty} e^{-x}x^4 dx$ | (1) $\frac{2}{15}$ |
| (b) $\int_0^{\pi/2} \sin^2 x \cos^3 x \, dx$ | (2) 24 |
| (c) $\int_0^1 x^6\sqrt{1-x^2}\, dx$ | (3) $5\pi/256$ |
| (d) $\left\lvert\frac{1}{4}\right.\left\lvert\frac{3}{4}\right.$ | (4) $\sqrt{2}\pi$ |

The correct match is

| | a | b | c | d |
|---|---|---|---|---|
| (i) | (1) | (2) | (3) | (4) |
| (ii) | (1) | (3) | (4) | (3) |
| (iii) | (2) | (1) | (3) | (4) |
| (iv) | (2) | (1) | (4) | (3) |

Ans. (iii)

13. Consider the Assertion (A) and Reason given below.

Assertion (A)

$\int_0^t \sin x\, dx = 1 - \cos t$

Reason (R) sin x is continuous in any closed interval [0, 1].

(I.A.S. 1993)

(i) Both A and R are true and R is the correct explanatation of A.
(ii) Both A and R are true but R is not a correct explanation of A.
(iii) A is true but R is false.
(iv) A is false but R is true.

Ans. (i)

# UNIT - 5
# Vector Calculus

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# Chapter 12
# Vector Differential Calculus

**Introduction:-** Let the vector $\vec{r}$ be a function of a scalar variable t, then

$\vec{r} = f(t)$

If only one value of $\vec{r}$ corresponds to each value of t, then $\vec{r}$ is defined as a single valued function of the scalar variable t. If t varies continuously, so does $\vec{r}$. As a result, the end point $\vec{r}$ describes a continuous curve.

The following illustration make the point clear :

$\vec{r} = a \cos t\ \hat{i} + a \sin t\ \hat{j}$ (Circle)

$\vec{r} = a \cos t\ \hat{i} + b \sin t\ \hat{j}$ (Ellipse)

$\vec{r} = at^2\ \hat{i} + 2\,at\ \hat{j}$ (Parabola)

$\vec{r} = a \sec t\ \hat{i} + b \tan t\ \hat{j}$ (Hyperbola)

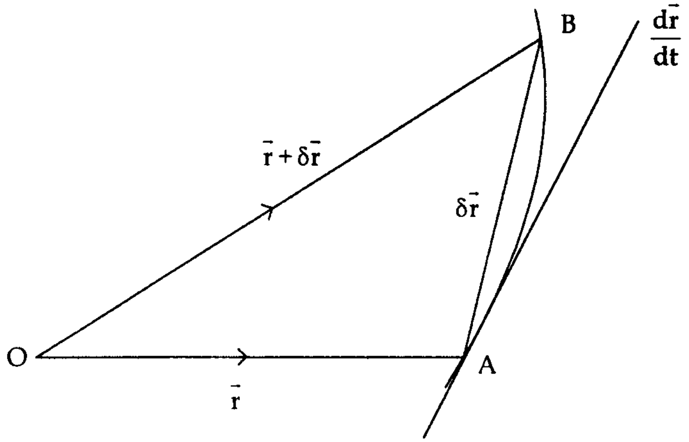
All these are the vector equations of the curves. For different values of t, the end point of the vector describes the curve as mentioned above.

The vector analysis consists of two parts (i) vector Algebra & (ii) Vector Calculus. Students have already studied vector algebra, so at present one shall study only the vector calculus which is very useful while solving problems of mechanics, fluid mechanics and other branches of Engineering and Technology.

**Differentiation of Vector :-**

Let the vector $\vec{r}$ be a continuous and single valued function of scalar variable t (i.e. length of the vector can be determined as soon as a value of t is given) with O as origin, let the vector $\vec{r}$ be represented by OA for a certain value of t and let $\vec{r} + \delta\vec{r}$ be represented by $\overrightarrow{OB}$ corresponding to the value $t + \delta t$ where $\delta t$ is a small increment in t. Then $\delta t$ produces an increment $(\vec{r} + \delta\vec{r} - \vec{r})$ i.e. $\delta\vec{r}$ in $\vec{r}$.

The increment $\delta\vec{r}$ is equal to $\overrightarrow{AB}$. There also the quotient $\frac{\partial \vec{r}}{\partial t}$ is a vector if $\delta t \to 0$ then $\delta\vec{r} \to 0$ and the point B moves towards A to coincide with it and then chord AB coincides with the tangent at P to the curve.



If the limiting value of the quotient $\frac{\overline{\delta r}}{\delta t}$ as δt→ 0 exists, then this value is defined as the differential coefficient of $\vec{r}$ with respect to t and the vector $\vec{r}$ is to be differentiable and is denoted by $\frac{d\vec{r}}{dt}$. This process is known as differentiation and the differential coefficient is known as the derivative or the derivative function.

Thus $\frac{d\vec{r}}{dt} = \lim_{\delta t\to 0} \frac{\overline{\delta r}}{\delta t} = \lim_{\delta t\to 0} \frac{(\vec{r}+\delta\vec{r})-\vec{r}}{\delta t}$

Since $\frac{d\vec{r}}{dt}$ is itself a vector function of t, its derivative is denoted by $\frac{d^2\vec{r}}{dt^2}$ and is called the second derivative of $\vec{r}$ with respect to t. Similarly, we can define higher order derivatives of $\vec{r}$.

**ILLUSTRATIVE EXAMPLES**

**Example1** : If $\vec{r} = a\cos t\ \hat{i} + a\sin t\ \hat{j} + t\ \hat{k}$ Find $\frac{d\vec{r}}{dt}, \frac{d^2\vec{r}}{dt^2}, \left|\frac{d^2\vec{r}}{dt^2}\right|$

**Solution** : we have

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}(a\cos t)\hat{i} + \frac{d}{dt}(a\sin t)\ \hat{j} + \frac{d}{dt}(t)\ \hat{k}$$

$$= -a\sin t\ \hat{i} + a\cos t\ \hat{j} + \hat{k}$$

$$\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}(-\,a \sin t)\ \hat{i} + \frac{d}{dt}(a \cos t)\ \hat{j} + \frac{d}{dt}(\hat{k})$$

$$= -\,a \cos t\ \hat{i} - a \sin t\ \vec{j}$$

Hence $\left|\dfrac{d^2\vec{r}}{dt^2}\right| = \sqrt{(-a \cos t)^2 + (-a \sin t)^2} = a$

**Example 2** : If $\dfrac{d\vec{A}}{dt} = \vec{C} \times \vec{A}$ and $\dfrac{d\vec{B}}{dt} = \vec{C} \times \vec{B}$, Prove that $\dfrac{d}{dt}(\vec{A} \times \vec{B}) = \vec{C} \times (\vec{A} \times \vec{B})$

(UPTU 2001)

**Solution** : $\dfrac{d}{dt}(\vec{A} \times \vec{B}) = \vec{A} \times \dfrac{d\vec{B}}{dt} + \dfrac{d\vec{A}}{dt} \times \vec{B}$

$= \vec{A} \times (\vec{C} \times \vec{B}) + (\vec{C} \times \vec{A}) \times \vec{B}$

$= (\vec{A}.\vec{B})\vec{C} - (\vec{A}.\vec{C})\vec{B} + (\vec{B}.\vec{C})\vec{A} - (\vec{B}.\vec{A})\vec{C}$

$\because \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

$= (\vec{B}.\vec{C})\vec{A} - (\vec{A}.\vec{C})\vec{B}$

$\because \vec{a}.\vec{b} = \vec{b}.\vec{a}$

$= (\vec{C}.\vec{B})\vec{A} - (\vec{C}.\vec{A})\vec{B}$

$= \vec{C} \times (\vec{B} \times \vec{A})$ hence proved.

**Example 3** : If $\vec{r} = (a \cos t)\ \hat{i} + (a \sin t)\ \vec{j} + (at \tan\alpha)\,\hat{k}$ then evaluate

$$\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| \text{ and } \left[\frac{d\vec{r}}{dt}, \frac{d^2\vec{r}}{dt^2}, \frac{d^3\vec{r}}{dt^3}\right]$$

**Solution** : Given $\vec{r} = (a \cos t)\ \hat{i} + (a \sin t)\ \vec{j} + (at \tan\alpha)\,\hat{k}$

$$\therefore \frac{d\vec{r}}{dt} = (-a \sin t)\hat{i} + (a \cos t)\hat{j} + (a \tan\alpha)\hat{k} \qquad \text{(i)}$$

$$\frac{d^2\vec{r}}{dt^2} = (-a \cos t)\hat{i} + (-a \sin t)\hat{j} + (O)\hat{k} \qquad \text{(ii)}$$

$$\frac{d^3\vec{r}}{dt^3} = (a \sin t)\hat{i} + (-a \cos t)\hat{j} \qquad \text{(iii)}$$

$$\therefore \frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$= a^2 \sin t \tan\alpha\ \hat{i} - a^2 \cos t \tan\alpha\ \hat{j} + a^2 \hat{k}$

$$\therefore \left|\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right| = \sqrt{(a^2 \sin t \tan\alpha)^2 + (-a^2 \cos t \tan\alpha)^2 + (a^2)^2}$$

$= \sqrt{a^4 \sin^2 t \tan^2\alpha + a^4 \cos^2 t \tan^2\alpha + a^4}$

$= a^2 \sqrt{\tan^2\alpha + 1}$

$= a^2 \sec\alpha$

Also we know $\left[\vec{a}, \vec{b}, \vec{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

where $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\therefore \left[\frac{d\vec{r}}{dt}, \frac{d^2\vec{r}}{dt^2}, \frac{d^3\vec{r}}{dt^3}\right] = \begin{vmatrix} -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}$$

from (i) (ii) and (iii)

$= a \tan\alpha\ (a^2 \cos^2 t + a^2 \sin^2 t)$ expanding the determinant

$= a^3 \tan\alpha$.

**Example 4** : If $\hat{r}$ is a unit vector show that

$$\left|\hat{r}\times\frac{d\hat{r}}{dt}\right| = \left|\frac{d\hat{r}}{dt}\right|$$

(I.A.S 1971)

**Solution** : We know $\frac{d\hat{r}}{dt}$ is a vector perpendicular to $\hat{r}$

$$\therefore \hat{r}\times\frac{d\hat{r}}{dt} = |\hat{r}|\left|\frac{d\hat{r}}{dt}\right| \sin 90^0 .\hat{n}$$

Where $\hat{n}$ is a unit vector perpendicular to $\hat{r}$ as well as $\frac{d\hat{r}}{dt}$ and $\hat{r}, \frac{d\hat{r}}{dt}$ and $\hat{n}$ form a right handed triad of vectors or $\hat{r} \times \frac{d\hat{r}}{dt} = \left|\frac{d\hat{r}}{dt}\right| \hat{n} \because |\hat{r}| = 1$, $\hat{r}$ being unit vector

or $\left|\hat{r} \times \frac{d\hat{r}}{dt}\right| = \left|\frac{d\hat{r}}{dt}\right| \therefore |\hat{n}| = 1$, $\hat{n}$ being unit vector

Hence proved.

**Example 5** :- A particle moves along the curve x = 4 cost, y = sin t, z= 6t Find the velocity and acceleration at time t = 0 and t = π/2

**Solution** :- The position vector of the particle at any time t is given by

$\vec{r} = x\hat{i} + y\vec{j} + z\hat{k}$

$\therefore$ i.e., $\hat{r} = (4\cos t)\,\hat{i} + (4\sin t)\,\vec{j} + 6t\,\hat{k}$

$\therefore$ velocity $\vec{v} = \frac{d\hat{r}}{dt} = (-4\sin t)\,\hat{i} + (4\cos t)\,\vec{j} + 6\hat{k}$ (i)

and acceleration

$\vec{a} = \frac{d^2\vec{r}}{dt^2} = -(4\cos t)\,\hat{i} - (4\sin t)\,\vec{j}$ (ii)

From (i) and (ii) we have

at $t = 0$, $\vec{v} = 4\vec{j} + 6\hat{k}$ and $\vec{a} = -4\hat{i}$

at $t = \pi/2$. $\vec{v} = -4\hat{i} + 6\hat{k}$ and $\vec{a} = -4\vec{j}$

**Example 6** : A particle moves along the carve

$x = t^3 + 1$, $y = t^2$, $z = 2t + 5$

Where t is the time. Find the components of its velocity and acceleration at time t = 1 in the direction $\hat{i} + \vec{j} + 3\hat{k}$.

**Solution** : Unit vector in the direction of $\hat{i} + \vec{j} + 3\hat{k}$ is

$$= \frac{\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{(1)^2 + (1)^2 + (3)^2}} = \frac{\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{11}} \qquad \text{(i)}$$

Now velocity $\frac{d\vec{r}}{dt} = \frac{d}{dt}(x\hat{i} + y\vec{j} + z\hat{k})$

$= \frac{d}{dt}\left\{(t^3 + 1)\hat{i} + t^2\hat{j} + (2t + 5)\hat{k}\right\}$

$= 3t^2\hat{i} + 2t\,\hat{j} + 2\hat{k}$

$= 3\hat{i} + 2\hat{j} + 2\hat{k}$ at t= 1 (ii)

Hence the component of velocity at t =1 in the direction of the vector $\hat{i} + \hat{j} + 3\hat{k}$ is

$=\left(3\hat{i}+2\hat{j}+2\hat{k}\right)$. (Unit vector in the direction of $\hat{i}+\hat{j}+3\hat{k}$)

$$=\left(3\hat{i}+2\hat{j}+2\hat{k}\right).\frac{\left(\hat{i}+\hat{j}+3\hat{k}\right)}{\sqrt{11}}$$

$$=\frac{3+2+6}{\sqrt{11}}=\sqrt{11}$$

Again, acceleration

$$=\frac{d^2\vec{r}}{dt^2}$$

$$=\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)$$

$$=\frac{d}{dt}\left(3t^2\hat{i}+2t\hat{j}+2\hat{k}\right)$$

$$=6t\hat{i}+2\vec{j}$$

$$=6\hat{i}+2\vec{j} \text{ at } t=1$$

Hence, the component of acceleration at t = 1 in the direction of the vector $\hat{i}+\hat{j}+3\hat{k}$ is

$=(6\hat{i}+2\vec{J})$. (Unit vector along $\hat{i}+\hat{j}+3\hat{k}$)

$$=\left(6\hat{i}+2\hat{j}\right).\frac{\left(\hat{i}+\hat{j}+3\hat{k}\right)}{\sqrt{11}}$$

$=\frac{6+2}{\sqrt{11}}=\frac{8}{\sqrt{11}}$ Answer.

**Example 7 :** Show that $\hat{r}\times d\hat{r}=\frac{\vec{r}\times d\vec{r}}{r^2}$ where $\hat{r}=\frac{\vec{r}}{r}$

**Solution :** Since $\hat{r}=\frac{\vec{r}}{r}$

$$\therefore d\hat{r}=d\left(\frac{\vec{r}}{r}\right)$$

$$=\frac{1}{r}d\vec{r}+\vec{r}\,d\left(\frac{1}{r}\right)$$

$$=\frac{1}{r}d\vec{r}+\vec{r}\left(-\frac{1}{r^2}dr\right)$$

Therefore,

$$\hat{r} \times d\hat{r} = \hat{r} \times \left(\frac{1}{r}d\vec{r} - \frac{\vec{r}}{r^2}dr\right)$$

$$= \frac{\vec{r}}{r} \times \left(\frac{1}{r}d\vec{r} - \frac{\vec{r}}{r^2}dr\right)$$

$$= \frac{\vec{r} \times d\vec{r}}{r^2} - \frac{\vec{r} \times \vec{r}}{r^3}dr$$

$$= \frac{\vec{r} \times d\vec{r}}{r^2} \qquad \because \vec{r} \times \vec{r} = \vec{0}$$

Hence proved.

**Example 8** : A particle P is moving on a circle of radius r with constant angular velocity $\omega = \frac{d\theta}{dt}$. Show that the acceleration is $-\omega^2 \vec{r}$.

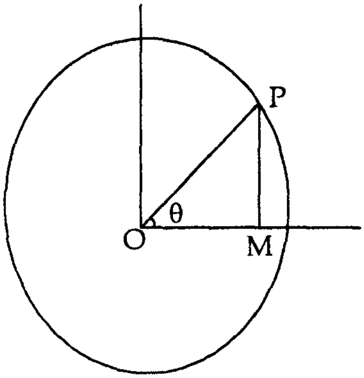
**Solution:-** Let $\hat{i}$ and $\vec{j}$ be the unit vectors along two perpendicular radii of the circle.

If P be any point on the circle such that OP makes an angle $\theta$ with $\hat{i}$ then the position vector of P is give by

$$\vec{r} = \overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP}$$

$$= (r\cos\theta)\,\hat{i} + (r\sin\theta)\,\vec{j}$$

Where r is the radius of circle and hence constant.



$$\frac{d\vec{r}}{dt} = -\,r\sin\theta\,\frac{d\theta}{dt}\,\hat{i} + r\cos\theta\,\frac{d\theta}{dt}\,\vec{j}$$

$$= (-r\sin\theta\,\hat{i} + r\cos\theta\,\vec{j}\,)\,\omega$$

As $\frac{d\theta}{dt} = \omega$

Hence $\vec{a} = \frac{d^2\vec{r}}{dt^2} = (-\, r\cos\theta \frac{d\theta}{dt}\,\hat{i} - r\sin\theta \frac{d\theta}{dt}\,\vec{j}\,)\,\omega$

$= -\,(r\cos\theta\,\hat{i} + r\sin\theta\,\vec{j}\,)\,\omega^2$

$= -w^2\,\vec{r}$

**TICK THE CORRECT ANSWER FROM THE CHOICES GIVEN BELOW**

(1) The necessary and sufficient condition for the vector function $\vec{a}(t)$ to be constant is $\frac{d\vec{a}}{dt}$

(i) $\vec{0}$ (ii) $\vec{a}$

(iii) $2\vec{a}$ (iv) 2a

Ans : (i)

(2) If $\vec{r} = \vec{a}\,\cos wt + \vec{b}\,\sin wt$, then the value of $\vec{r} \times \frac{d\vec{r}}{dt}$ is

(i) $w^2\vec{r}$ (ii) $-w^2\vec{r}$

(iii) $w\,\vec{a} \times \vec{b}$ (iv) $w\,\vec{b} \times \vec{a}$

Ans : (iii)

(3) If $\vec{r} = \vec{a}\,\cos wt + \vec{b}\,\sin wt$, then the value of $\frac{d^2\vec{r}}{dt^2}$ is

(i) $w^2\vec{r}$ (ii) $-w^2\vec{r}$

(iii) $w\,\vec{a} \times \vec{b}$ (iv) $w\,\vec{b} \times \vec{a}$

Ans : (ii)

(4) If $\vec{a}$ is a vector function of some scalar t such that $\vec{a}$ has constant magnitude, then $\vec{a}.\frac{d\vec{a}}{dt}$ is

(i) 0 (ii) $\vec{a}$

(iii) $2\vec{a}$ (iv) 2a

Ans : (i)

(5) If the direction of a vector function $\vec{a}(t)$ is constant, then $\vec{a} \times \frac{d\vec{a}}{dt}$ is

(i) $\vec{0}$ (ii) $\vec{a}$

(iii) $2\vec{a}$ (iv) 2a
Ans : (i)

(6) If $\vec{r} = \vec{a}\,e^{nt} + \vec{b}\,e^{-nt}$, where $\vec{a}, \vec{b}$ are constant vectors then $\frac{d^2\vec{r}}{dt^2}$ is

(i) $n^2\vec{r}$ (ii) $-n^2\vec{r}$
(iii) 0 (iv) $n\vec{r}$
Ans : (i)

(7) If $\vec{r} = \sin t\,\hat{i} + \cos t\,\vec{j} + t\hat{k}$ then $\left|\frac{d^2\vec{r}}{dt^2}\right|$ is

(i) 0 (ii) 1
(iii) 2 (iv) 3
Ans : (ii)

(8) A particle moves along the curve $x=e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$, where t is the time. The magnitudes of the velocity and acceleration at t = 0 is

(i) $|\vec{v}| = \sqrt{37}$, $|\vec{a}| = \sqrt{325}$

(ii) $|\vec{v}| = \sqrt{325}$, $|\vec{a}| = \sqrt{37}$

(iii) $|\vec{v}| = \sqrt{\frac{37}{325}}$, $|\vec{a}| = \sqrt{\frac{325}{37}}$

(iv) $|\vec{v}| = \sqrt{\frac{1}{37}}$, $|\vec{a}| = \sqrt{\frac{1}{325}}$

Ans : (i)

(9) If $\vec{r} = (a\cos t)\,\hat{i} + (a\sin t)\,\vec{j} + t\hat{k}$ then $\left|\frac{d^2\vec{r}}{dt^2}\right|$ is

(i) 0 (ii) a
(iii) 2a (iv) 3a
Ans : (ii)

(10) If $\vec{U} = t^2\hat{i} - t\,\vec{j} + (2t+1)\,\hat{k}$ and $\vec{V} = (2t-3)\,\hat{i} + \vec{j} - t\hat{k}$ then $\frac{d}{dt}(\vec{U}\cdot\vec{V})$ when t = 1 is

(i) 0 (ii) 2
(iii) -4 (iv) -6

Ans : (iv)

(11) $\vec{A} = t^m \vec{a} + t^n \vec{b}$, When $\vec{a}, \vec{b}$ are constant vectors, show that, if $\vec{A}$ and $\frac{d^2\vec{A}}{dt^2}$ are parallel vectors then m + n is

(i) 1 (ii) -1

(iii) 2 (iv) 0

Ans : (i)

**The differential Operator Del (∇)**

The operator ∇ is defined as

$$\nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

∇is also known as nabla. It behave as a vector.

## SCALAR AND VECTOR POINT FUNCTIONS (UPTU 2001)

A variable quantity whose value at any point in a region of space depends upon the position of the point, is called a point function. There are two types of point functions.

(i) **Scalar point function:** Let R be a region of space at each point of which a scalar $\phi = \phi(x,y,z)$ is given, then $\phi$ is called a scalar function and R is called a scalar field.

The temperature distribution in a medium, the distribution of atmospheric pressure in space are examples of scalar point functions.

**(ii) Vector point function :** Let R be a region of space at each point of which a vector $\vec{V} = \vec{V}(x, y, z)$ is given them $\vec{V}$ is called a vector point function and R is called a vector field. Each vector $\vec{V}$ of the field is regarded as a localized vector attached to the corresponding point (x, y, z).

The velocity of a moving fluid at any instant, the gravitational force are examples of vector point function.

## GRADIENT OF A SCALAR POINT FUNCTION :

If $\phi$ (x, y, z) is a scalar point function and continuously differentiable then the gradient of $\phi$ is defined as grad $\phi = \nabla\phi \equiv \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$

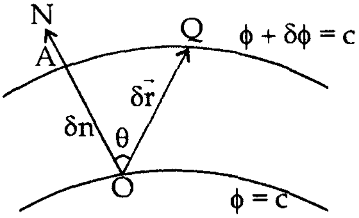
$$= \Sigma \hat{i}\frac{\partial \phi}{\partial x}$$

Thus $\nabla\phi$ is a vector whose rectangular components are

$\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}$ and $\frac{\partial\phi}{\partial z}$

**Geometrical Interpretation of Gradient :**

If a surface $\phi\,(x, y, z) = c$ is drawn through any point P such that at each point on the surface, the function has the same value as at P, then such a surface is called a level surface through P, for example, if $\phi\,(x, y, z)$ represents potential at the point $(x, y, z)$, the equipotential surface $\phi\,(x, y, z) = c$ is a level surface.



Through any point passes one and only one level surface. Moreover, no two level surfaces can intersect.

Consider the level surface through P at which the function has value $\phi$ and another level surface through a neighbouring point Q where the value is $\phi + \delta\phi$.

Let $\vec{r}$ and $\vec{r} + \delta\vec{r}$ be the position vectors of P and Q respectively, then $\overrightarrow{PQ} = \delta\vec{r}$

$$\text{Now } \nabla\phi.\delta\vec{r} = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right).\left(\hat{i}\delta x + \hat{j}\delta y + \hat{k}\delta z\right)$$

$$= \frac{\partial\phi}{\partial x}\delta x + \frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial z}\delta z$$

$$= \delta\phi \qquad \text{(i)}$$

If Q lies on the same level surface as P, then $\delta\phi = 0$

$\therefore$ (i) reduces to $\nabla\phi.\delta\vec{r} = 0$

Thus $\nabla\phi$ is perpendicular to every $\delta\vec{r}$ lying in the surface.

Hence $\nabla\phi$ is normal to the surface $\phi\,(x, y, z) = C$

Let $\nabla\phi = |\nabla\phi|\hat{N}$, Where $\hat{N}$ is a unit vector normal to the surface. Let PA=$\delta n$ be the perpendicular distance between the two level surfaces through P and Q. Then the rate of change of $\phi$ in the direction of normal to the surface through P is

$$\frac{\partial\phi}{\partial n} = \lim_{\delta n\to 0}\frac{\delta\phi}{\delta n} = \lim_{\delta n\to 0}\frac{\nabla\phi.\delta\bar{r}}{\delta n} \text{ by (i)}$$

$$= \lim_{\delta n\to 0}\frac{|\nabla\phi|\,|\hat{N}.\delta\bar{r}|}{\delta n} = |\nabla\phi|$$

$$\because \hat{N}\cdot\delta\vec{r} = |\hat{N}||\delta\vec{r}|\cos\theta$$

$$= |\delta\vec{r}|\cos\theta$$

$$= \delta n$$

$$\therefore |\nabla\phi| = \frac{\partial\phi}{\partial n}$$

Hence the gradient of a scalar field $\phi$ is a vector normal to the surface $\phi = c$ and has a magnitude equal to the rate of change of $\phi$ along this normal.

**Directional Derivative :**

Let $\phi$ (x, y, z) be a scalar point function and s represent a distance from any point P(x, y, z) in the direction of a unit vector $\hat{a}$, then $\frac{d\phi}{ds}$ is called the directional derivative of $\phi$ in the direction of $\hat{a}$.

The directional derivative of a scalar point function is a scalar and of a vector point function is a vector.

**Theorem :-** The directional derivative of a scalar field $\phi$ at a point P(x, y, z) in the direction of a unit vector $\hat{a}$ is given by

$$\frac{d\phi}{ds} = \hat{a}.\text{grad}\,\phi$$

$$= \hat{a}.\nabla\phi$$

**Proof :-** Since $\hat{a}$ is a unit vector at the point P(x, y, z), therefore

$$.\hat{a} = \hat{i}\frac{dx}{ds} + \hat{j}\frac{dy}{ds} + \hat{k}\frac{dz}{ds}$$

Where s represents a distance from P in the direction of $\hat{a}$.

Now $\hat{a}.\text{grad}\,\phi = \hat{a}.\nabla\phi$

$$= \left(\hat{i}\frac{dx}{ds} + \hat{j}\frac{dy}{ds} + \hat{k}\frac{dz}{ds}\right).\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{dx}{ds}.\frac{\partial\phi}{\partial x} + \frac{dy}{ds}.\frac{\partial\phi}{\partial y} + \frac{dz}{ds}.\frac{\partial\phi}{\partial z}$$

$$= \frac{d\phi}{ds} \quad \text{Hence Proved.}$$

**Example 1:** If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ show that

(i) $\nabla r = \hat{r}$ (ii) $\nabla\frac{1}{r} = -\frac{\hat{r}}{r^2}$

(iii) $\nabla r^n = nr^{n-2}\,\vec{r}$

(UPTU 2007)

**Solution :** - Since $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\Rightarrow |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow r = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow r^2 = x^2 + y^2 + z^2$$

(i) $$\nabla r = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) r$$

$$= \hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}$$

$$= \hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}$$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}$$

$$= \frac{\vec{r}}{r}$$

$$= \hat{r}$$

$\because r^2 = x^2 + y^2 + z^2$

$$\Rightarrow 2r\frac{\partial r}{\partial x} = 2x$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$

and $\frac{\partial r}{\partial z} = \frac{z}{r}$

(ii) $$\nabla \frac{1}{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\frac{1}{r}$$

$$= \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= -i\frac{1}{r^2}\frac{\partial r}{\partial x} - j\frac{1}{r^2}\frac{\partial r}{\partial y} - k\frac{1}{r^2}\frac{\partial r}{\partial z}$$

$$= -\frac{1}{r^2}\left(\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right)$$

$$= -\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r^3}$$

$$= -\frac{\vec{r}}{r^3}$$

$$= -\frac{\hat{r}}{r^2}$$

$\because \hat{r} = \frac{\vec{r}}{r}$

(iii) $\nabla r^n = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)r^n$

$= \hat{i}\,nr^{n-1}\frac{\partial r}{\partial x}+\hat{j}\,nr^{n-1}\frac{\partial r}{\partial y}+\hat{k}\,nr^{n-1}\frac{\partial r}{\partial z}$

$= \hat{i}\,nr^{n-1}\frac{x}{r}+\hat{j}\,nr^{n-1}\frac{y}{r}+\hat{k}\,nr^{n-1}\frac{z}{r}$

$= \hat{i}\,nr^{n-2}x+\hat{j}\,nr^{n-2}y+\hat{k}\,nr^{n-2}z$

$= nr^{n-2}\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$

$= nr^{n-2}\vec{r}$

**Example 2:-** If $\phi\,(x, y, z) = 3x^2y - y^3z^2$, find $\nabla\phi$ at the point (1, -2, -1)

**Solution:-** $\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(3x^2y - y^3z^2\right)$

$$= \hat{i}\frac{\partial}{\partial x}\left(3x^2y - y^3z^2\right)+\hat{j}\frac{\partial}{\partial y}\left(3x^2y - y^3z^2\right)+\hat{k}\frac{\partial}{\partial z}\left(3x^2y - y^3z^2\right)$$

$$= \hat{i}\,(6xy)+\hat{j}\left(3x^2 - 3y^2z^2\right)+\hat{k}\left(-2y^3z\right)$$

$\therefore$ At (1,-2, -1) we have

$\nabla\phi = \vec{i}\ [6\,(1)\,(-2)] + \vec{j}\ [3(1)^2-3(-2)^2\,(-1)^2] + \vec{k}\ [-2\,(-2)^3\,(-1)]$

$= -12\vec{i} - 9\vec{j} - 16\vec{k}$ Answer

**Example 3 :-** Show that $\nabla(\vec{a}.\vec{r}) = \vec{a}$ where $\vec{a}$ is a constant vector.

**Proof :** Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

Then $\vec{a}.\vec{r} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}).\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$

$= a_1x + a_2y + a_3z$

Therefore $\nabla\left(\vec{a}.\vec{r}\right) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(a_1x + a_2y + a_3z\right)$

$= \hat{i}\,a_1 + \hat{j}\,a_2 + \hat{k}\,a_3$

$= \vec{a}$, hence proved.

**Example 4 :** If $\vec{a}$ and $\vec{b}$ be constant vectors, then show that grad $[\vec{r}\ \vec{a}\ \vec{b}] = \vec{a}\times\vec{b}$

**Solution :-** Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

Then $\phi = \left[\vec{r}\,\vec{a}\,\vec{b}\right] = \begin{vmatrix} x & y & z \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$= x(a_2b_3 - a_3b_2) + y(a_3b_1 - a_1b_3) + z(a_1b_2 - a_2b_1)$

Therefore grad $\phi$ = grad $\left[\vec{r}\,\vec{a}\,\vec{b}\right] = \hat{i}\dfrac{\partial\phi}{\partial x} + \hat{j}\dfrac{\partial\phi}{\partial y} + \hat{k}\dfrac{\partial\phi}{\partial z}$

$= \hat{i}(a_2b_3 - a_3b_2) + \hat{j}(a_3b_1 - a_1b_3) + \hat{k}(a_1b_2 - a_2b_1)$

$\because \dfrac{\partial\phi}{\partial x} = a_2b_3 - a_3b_2$

$\dfrac{\partial\phi}{\partial y} = a_3b_1 - a_1b_3$

and $\dfrac{\partial\phi}{\partial z} = a_1b_2 - a_2b_1$

$= \vec{a} \times \vec{b}$ Hence proved

**Example 5 :** If $\phi(x, y) = \log\sqrt{x^2 + y^2}$ show that

$$\text{grad } \phi = \frac{\vec{r} - (\vec{k}.\vec{r})\vec{k}}{\left\{\vec{r} - (\vec{k}.\vec{r})\vec{k}\right\}.\left\{\vec{r} - (\vec{k}.\vec{r})\vec{k}\right\}}$$

**Solution :** - we have $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ (i)

Therefore $\vec{r}.\vec{k} = z$ (ii)

Now $\phi = \dfrac{1}{2}\log(x^2 + y^2)$

$\because \dfrac{\partial\phi}{\partial x} = \dfrac{1}{2(x^2 + y^2)}.2x = \dfrac{x}{x^2 + y^2}$

similarly $\dfrac{\partial\phi}{\partial y} = \dfrac{y}{x^2 + y^2}, \dfrac{\partial\phi}{\partial z} = 0$

Thus grad $\phi = \vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z}$

$= \dfrac{x}{x^2 + y^2}\vec{i} + \dfrac{x}{x^2 + y^2}\vec{j} + 0\vec{k}$

$$=\frac{x\vec{i}+y\vec{j}}{x^2+y^2}$$

$$=\frac{\vec{r}-z\vec{k}}{\left(x\vec{i}+y\vec{j}\right).\left(x\vec{i}+y\vec{j}\right)}$$

$$=\frac{\vec{r}-z\vec{k}}{\left(\vec{r}-z\vec{k}\right).\left(\vec{r}-z\vec{k}\right)}$$ by (i)

Now by replacing z by $\vec{r}\,.\,\vec{k}$ we get

$$\text{grad } \phi=\frac{\vec{r}-\left(\vec{k}.\vec{r}\right)\vec{k}}{\left\{\vec{r}-\left(\vec{k}.\vec{r}\right)\vec{k}\right\}.\left\{\vec{r}-\left(\vec{k}.\vec{r}\right)\vec{k}\right\}}$$

Hence proved.

**Example 6 :** Find the directional derivative of $\phi = xy + yz + zx$ in the direction of vector $\hat{i}+2\hat{j}+2\hat{k}$ at (1, 2, 0)

**Solution :-** Since we know

directional derivative = $\hat{a}$ . grad $\phi$

Now grad $\phi=\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}$

$= (y + z)\ \hat{i} + (z + x)\ \hat{j} + (x + y)\ \hat{k}$

$= 2\hat{i}+\hat{j}+3\hat{k}$ at (1, 2, 0)

Also $\hat{a}=\frac{\hat{i}+2\hat{j}+2\hat{k}}{3}$

$\therefore$directional derivative $=\frac{1}{3}\ (\hat{i}+2\hat{j}+2\hat{k}).(2\hat{i}+\hat{j}+3\hat{k})$

$=\frac{1}{2}(2+2+6)$

$=\frac{10}{3}$ Answer.

**Example 7:** Find the directional derivative of the function $\phi = x^2 - y^2 + 2z^2$ at the point P (1, 2, 3) in the direction of the line PQ, where Q is the point (5, 0, 4).

(UPTU 2000).

**Solution :-** The position vectors of the points P and Q are respectively

$\hat{i}+2\hat{j}+3\hat{k}$ and $5\hat{i}+4\hat{k}$

$\overrightarrow{PQ} = (5\hat{i}+4\hat{k}) - (\hat{i}+2\hat{j}+3\hat{k})$

$= 4\hat{i} - 2\hat{j} + \hat{k}$

The unit vector $\hat{a}$ along PQ is given by

$$\hat{a} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{(4)^2 + (-2)^2 + (1)^2}} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}$$

Now grad $\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$

$= 2x\,\hat{i} - 2y\,\hat{j} + 4z\,\hat{k}$

directional derivative = $\hat{a}$ . grad $\phi$

$$= \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}} . (2x\,\hat{i} - 2y\,\hat{j} + 4z\,\hat{k})$$

$$= \frac{8x + 4y + 4z}{\sqrt{21}}$$

$$= \frac{8.1 + 4.2 + 4.3}{\sqrt{21}} \text{ at } (1, 2, 3)$$

$= 4\sqrt{\frac{7}{3}}$ Answer.

**Example 8:-** Find the directional derivative of $\phi = x^2 - 2y^2 + 4z^2$ at (1, 1, -1) in the direction $2\hat{i} + \hat{j} - \hat{k}$. In what direction is the directional derivative from the point (1, 1, -1) is maximum and what is its value?

**Solution :-** We have, directional derivative in the direction of $\hat{a}$ is $\hat{a}$ . grad $\phi$

Hence grad $\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$

$= 2x\,\hat{i} - 4y\,\hat{j} + 8z\,\hat{k}$

$= 2\,\hat{i} - 4\,\hat{j} - 8\,\hat{k}$ at (1, 1, -1)

and $\hat{a} = \frac{2\hat{i} + \hat{j} - \hat{k}}{\sqrt{6}}$

Therefore, directional derivative

$$= \left(\frac{2\hat{i} + \hat{j} - \hat{k}}{\sqrt{6}}\right).\left(2\hat{i} - 4\hat{j} - 8\hat{k}\right)$$

$$= \frac{4 - 4 + 8}{\sqrt{6}} = \frac{8}{\sqrt{6}}$$

Again, directional derivative is maximum along the normal i.e. along grad $\phi$ i.e. $2\hat{i} - 4\hat{j} - 8\hat{k}$ and hence the maximum value of directional derivative is

$|\text{grad}\phi| = \sqrt{4+16+64} = 2\sqrt{21}$ Answer.

**Example 9 :-** Find the values of the constants a, b, c so that the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at (1, 2, -1) has a maximum magnitude 64 in a direction parallel to z-axis.

(IAS 2002, 2006)

**Solution :-** We know that the directional derivative is maximum along the normal i.e. along grad $\phi$

here, we have

$\text{grad}\,\phi = (ay^2 + 3cz^2x^2)\,\hat{i} + (2\,axy + bz)\,\hat{j} + (by + 2czx^3)\,\hat{k}$

$= (4a + 3c)\,\hat{i} + (4a - b)\,\hat{j} + (2b - 2c)\,\hat{k}$ at (1, 2, -1)

But directional derivative is maximum along z-axis. Hence the coefficients of $\hat{i}$ and $\hat{j}$ should be zero.

$\therefore$ 4a + 3c= 0 and 4a - b = 0

$\therefore \text{grad}\,\phi = (2b - 2c)\,\hat{k}$

Also maximum value of directional derivative = $|\text{grad}\phi|$

$\therefore 64 = 2(b - c)$

$\Rightarrow b - c = 32$

Solving these three equations, we get

a= 6, b = 24, c = -8 Answer.

**Problem 10 :-** Find the angle between the normals to the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point (2, -1, 2)

(U.P.T.U 2002, I.A.S 1973)

**Solution :-** Let the given surfaces represented by

$\phi_1 \equiv x^2 + y^2 - z - 3$ and $\phi_2 \equiv x^2 + y^2 + z^2 - 9$

$$\therefore \vec{n}_1 = \nabla\phi_1 = \hat{i}\frac{\partial\phi_1}{\partial x} + \hat{j}\frac{\partial\phi_1}{\partial y} + \hat{k}\frac{\partial\phi_1}{\partial z}$$

$= 2x\vec{i} + 2y\vec{j} - \vec{k}$

$= 4\vec{i} - 2\vec{j} - \vec{k}$ at the point (2, -1, 2) similarly, we have

$\vec{n}_2 = \nabla\phi_2 = 4\vec{i} - 2\vec{j} + 4\vec{k}$ at the point (2, -1, 2)

If $\vec{n}_1$ and $\vec{n}_2$ be the normals to the surfaces $\phi_1$ and $\phi_2$, then $\vec{n}_1 = \nabla\phi_1$ and $\vec{n}_2 = \nabla\phi_2$

Let $\theta$ be the angle between the normals to the surface at the given point then

$\vec{n}_1.\vec{n}_2 = |\vec{n}_1|\,|\vec{n}_2|\cos\theta$

$$\therefore \cos\theta = \frac{\vec{n_1}.\vec{n_2}}{\left|\vec{n_1}\right|\left|\vec{n_2}\right|}$$

$$= \frac{\left(4\vec{i}-2\vec{j}-\vec{k}\right)}{\sqrt{16+4+1}}.\frac{\left(4\vec{i}-2\vec{j}+4\vec{k}\right)}{\sqrt{16+4+16}}$$

$$= \frac{16+4-4}{6\sqrt{21}}$$

$$= \frac{8}{3\sqrt{21}}$$

$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$ Answer.

**Problem 11 :-** Find the directional derivative of a function $\phi \equiv x^2\ y^3\ z^4$ at the point (2, 3, -1) in the direction making equal angles with the positive x, y, & z axis.

**Solution :-** Given $\phi = x^2y^3z^4$

Now grad $\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$

$= 2xy^3z^4\,\hat{i} + 3x^2y^2z^4\,\hat{j} + 4x^2y^3z^3\,\hat{k}$

If $\hat{a}$ be the unit vector in the required direction and $\alpha$ be the angle which $\hat{a}$ makes with the axes, then

$\hat{a} = (\cos\alpha)\,\hat{i} + (\cos\alpha)\,\hat{j} + (\cos\alpha)\,\hat{k}$

where $\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$

which gives $\cos\alpha = \frac{1}{\sqrt{3}}$

$\therefore \hat{a} = \frac{1}{\sqrt{3}}\left(\hat{i}+\hat{j}+\hat{k}\right)$

$\therefore$ directional derivative = $\hat{a}$ . grad $\phi$

$= \frac{1}{\sqrt{3}}\,(\hat{i}+\hat{j}+\hat{k}).(2xy^2z^4\,\hat{i} + 3x^2y^2z^4\,\hat{j} + 4x^2y^3z^3\,\hat{k})$

$= \frac{1}{\sqrt{3}}\,(2xy^3z^4+3x^2y^2z^4+4x^2y^3z^3)$

$= \frac{1}{\sqrt{3}}\,(108 + 108 - 432)$ at the point (2, 3, -1)

$= -\frac{216}{\sqrt{3}}$ Answer.

**Example 12 :-** Find a unit vector which is perpendicular to the surface of the paraboloid of revolution.
$z = x^2+y^2$ at the point (1, 2, 5)

(B.P.S.C 1997)

**Solution :-** $\phi = x^2+y^2-z$

$$\therefore \text{grad } \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

$$= \hat{i}.2x + \hat{j}.2y - \hat{k}.1$$

$$= 2\hat{i} + 4\hat{j} - \hat{k} \text{ at point } (1, 2, 5)$$

Hence unit normal $= \dfrac{\text{grad}\phi}{|\text{grad}\phi|}$

$$= \frac{2\hat{i}+4\hat{j}-\hat{k}}{\sqrt{4+16+1}}$$

$$= \frac{2\hat{i}+4\hat{j}-\hat{k}}{\sqrt{21}} \text{ Answer.}$$

**Problem 13 :** What is the greatest rate of increase of $\phi \equiv xyz^2$ at the point (1, 0, 3).

**Solution :** $\text{grad } \phi = \hat{i}\dfrac{\partial \phi}{\partial x} + \hat{j}\dfrac{\partial \phi}{\partial y} + \hat{k}\dfrac{\partial \phi}{\partial z}$

$$= \hat{i}\,yz^2 + \hat{j}\,xz^2 + \hat{k}\,2xyz$$

$$= \hat{i}\,0 + \hat{j}\,9 + \hat{k}\,0 \text{ at } (1, 0, 3)$$

$$= 9\hat{j}$$

Since we know the greatest rate of increase of $\phi = |\nabla \phi|$

$$= \sqrt{(9)^2}$$

= 9 Answer.

**Divergence of a vector :-**

If $\vec{V}$ (x, y, z) is any continuously differentiable vector point function, then the divergence of $\vec{V}$, written as div $\vec{V}$ or $\nabla . \vec{V}$ is defined by

$$\text{div } \vec{V} = \nabla \cdot \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\vec{V}$$

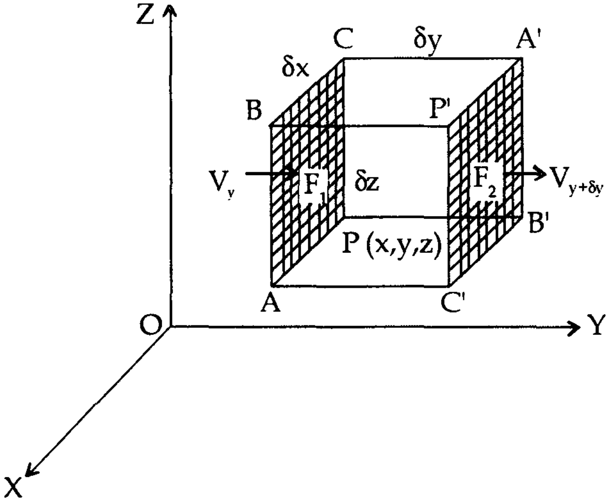
$$= \hat{i}\cdot\frac{\partial \vec{V}}{\partial x} + \hat{j}\cdot\frac{\partial \vec{V}}{\partial y} + \hat{k}\cdot\frac{\partial \vec{V}}{\partial z}$$

$$= \Sigma \hat{i}\cdot\frac{\partial \vec{V}}{\partial x}$$

$\nabla.\vec{V}$ is a scalar product of the operator $\nabla$with the vector $\vec{V}$.

**Physical Interpretation of Divergence**

Consider a fluid having density $\rho = \rho\,(x, y, z, t)$ and velocity $\vec{v} = \vec{v}\,(x, y, z, t)$ at a point (x,y,z) at time t. Let $\vec{V} = \rho\,\vec{v}$, then $\vec{V}$ is a vector having the same direction as $\vec{v}$ and magnitude $\rho|\vec{v}|$. It is known as flux. Its direction gives the direction of the fluid flow, and its magnitude gives the mass of the fluid crossing per unit time a unit area placed perpendicular to the direction of flow.



Consider the motion of the fluid having velocity $\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$ at point P(x, y, z). Consider a small parallelopiped with edges δx, δy, δz parallel to the axes with one of its corners at P. The mass of the fluid entering through the face $F_1$ per unit time is $V_y\,\delta x\,\delta z$ and that flowing out through the opposite face $F_2$ is $V_{y+\delta y}\,\delta x\,\delta z = \left(V_y + \frac{\partial V_y}{\partial y}\delta y\right)\delta x\,\delta z$ by using Taylor's Theorem.

∴The net decrease in the mass of fluid flowing across these two faces $= \left(V_y + \frac{\partial V_y}{\partial y}\delta y\right)\delta x\,\delta z - V_y\,\delta x\,\delta z$

$$=\frac{\partial V_y}{\partial y}\ \delta x\ \delta y\delta z$$

similarly, considering the other two pairs of faces, we get the total decrease in the mass of fluid inside the parallelopiped per unit time $\left(\frac{\partial \vec{V}x}{\partial x}+\hat{j}\frac{\partial \vec{V}y}{\partial y}+\hat{k}\frac{\partial \vec{V}z}{\partial z}\right)\delta x\delta y\delta z.$

Dividing this by the volume $\delta x\ \delta y\ \delta z$ of the parallelopiped, we have the rate of loss of fluid per unit time $=\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}$

$= \text{div}\,\vec{V}$

Hence div $\vec{V}$ gives the rate of outflow per unit volume at a point of the fluid.

If the fluid is incompressible, there can be no gain or loss in the volume element. Hence div $\vec{V} = 0$ and $\vec{V}$ is called a solenoidal vector function. Which in known in Hydrodynamics as the equation of continuity for incompressible fluids.

**Note** : Vectors having zero divergence are called solenoidal and are useful in various branches of physics and Engineering.

(U.P.T.U 2002, 2003, 2006).

## CURL OF VECTOR POINT FUNCTION

The curl (or rotation) of a differentiable vector point function $\vec{V}$ is denoted by curl $\vec{V}$ and is defined as

$$\text{curl}\ \vec{V}=\nabla\times\vec{V}=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\vec{V}$$

$$=\hat{i}\times\frac{\partial\vec{V}}{\partial x}+\hat{j}\times\frac{\partial\vec{V}}{\partial y}+\hat{k}\times\frac{\partial\vec{V}}{\partial z}$$

$$=\sum\hat{i}\times\frac{\partial\vec{V}}{\partial x}$$

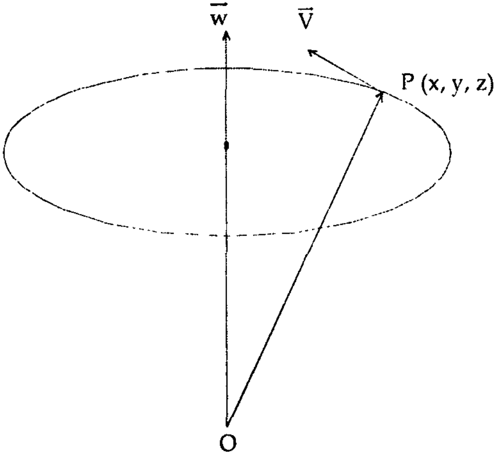
The curl of a vector point function is a vector quantity if $\vec{V}= V_1\hat{i}+ V_2\hat{j}+ V_3\hat{k}$ Then

$$\text{curl}\ \vec{V}=\nabla\times\vec{V}=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\left(V_1\hat{i}+V_2\hat{j}+V_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

**PHYSICAL INTERPRETATION OF CURL :-** Consider a rigid body rotating about a given axis through O with uniform angular velocity ω.



Let $\vec{\omega} = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}$

The linear velocity $\vec{V}$ of any point P(x, y, z) on the rigid body is given by

$\vec{V} = \vec{\omega} \times \vec{r}$

Where $\vec{r} = \hat{i}\,x + \hat{j}\,y + \hat{k}\,z$ is the position vector of P

$\therefore \vec{V} = \vec{\omega} \times \vec{r}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \hat{i}(\omega_2 z - \omega_3 y) + \hat{j}(\omega_3 x - \omega_1 z) + \hat{k}(\omega_1 y - \omega_2 x)$$

$\because$ curl $\vec{V}$ = curl $(\vec{\omega} \times \vec{r}) = \nabla \times (\vec{\omega} \times \vec{r})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= (\omega_1 + \omega_1)\hat{i} + (\omega_2 + \omega_2)\hat{j} + (\omega_3 + \omega_3)\hat{k}$$

$$= 2\left(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}\right)$$

$\therefore$ $\omega_1, \omega_2, \omega_3$ are constants

$$= 2\,\vec{\omega}$$

$$\therefore \vec{\omega} = \frac{1}{2}\,\text{curl}\,\vec{V}$$

Thus the angular velocity at any points is equal to half the curl of linear velocity at that point of the body.

(U.P.T.U 2001).

**Note** : If curl $\vec{V} = 0$, then $\vec{V}$ is said to be an irrotational vector, otherwise rotational. Also curl of a vector signifies rotation.

**VECTOR IDENTITIES**

(1) grad $\left(\vec{a}.\vec{b}\right) = \left(\vec{a}.\nabla\right)\vec{b} + \left(\vec{b}.\nabla\right)\vec{a} + \vec{a} \times \text{curl}\ \vec{b} + \vec{b} \times \text{curl}\ \vec{a}$

where $\vec{a}$ and $\vec{b}$ are the differentiable vector functions

(I.A.S 2004, U.P.P.C.S 2004)

**Proof :**

$$\text{grad}\left(\vec{a}.\vec{b}\right) = \hat{i}\frac{\partial}{\partial x}\left(\vec{a}.\vec{b}\right) + \hat{j}\frac{\partial}{\partial y}\left(\vec{a}.\vec{b}\right) + \hat{k}\frac{\partial}{\partial z}\left(\vec{a}.\vec{b}\right)$$

$$= \Sigma\hat{i}\left(\frac{\partial\vec{a}}{\partial x}\cdot\vec{b} + \vec{a}\cdot\frac{\partial\vec{b}}{\partial x}\right)$$

$$= \Sigma\hat{i}\left(\vec{b}\cdot\frac{\partial\vec{a}}{\partial x}\right) + \Sigma\hat{i}\left(\vec{a}\cdot\frac{\partial\vec{b}}{\partial x}\right) \qquad \text{(i)}$$

Again $\vec{a}\times\left(\hat{i}\times\dfrac{\partial\vec{b}}{\partial x}\right) = \left(\vec{a}.\dfrac{\partial\vec{b}}{\partial x}\right)\hat{i} - \left(\vec{a}.\hat{i}\right)\dfrac{\partial\vec{b}}{\partial x}$

or $\left(\vec{a}\cdot\dfrac{\partial\vec{b}}{\partial x}\right)\hat{i} = \vec{a}\times\left(\hat{i}\times\dfrac{\partial\vec{b}}{\partial x}\right) + \left(\vec{a}.\hat{i}\right)\dfrac{\partial\vec{b}}{\partial x}$

$$\therefore \sum\left(\vec{a}\cdot\frac{\partial \vec{b}}{\partial x}\right)\hat{i} = \vec{a}\times\left(\sum\hat{i}\times\frac{\partial\vec{b}}{\partial x}\right)+\sum\left(\vec{a}\,.\,\hat{i}\frac{\partial}{\partial x}\right)\vec{b}$$

$= \vec{a}\times \text{curl } \vec{b}+(\vec{a}\,.\nabla)\ \vec{b}$ (ii)

similarly

$\sum\left(\vec{b}\cdot\frac{\partial \vec{a}}{\partial x}\right)\hat{i} = \vec{b}\times \text{curl } \vec{a}+(\vec{b}\,.\nabla)\ \vec{a}$ ............(iii)

Hence from (i) , (ii) & (iii) we get

grad $\left(\vec{a}.\vec{b}\right)=\left(\vec{a}.\nabla\right)\vec{b}+\left(\vec{b}.\nabla\right)\vec{a}+\vec{a}\ \times \text{curl } \vec{b}+\ \vec{b}\ \times \text{curl}\,\vec{a}$

Hence proved.

(2) If $\vec{a}$ is a vector function and u is a scalar function then

div $(u\,\vec{a}) = u$ div $\vec{a}$ + (grad u) . $\vec{a}$

(U.P.T.U 2004, B.P.S.C 1995)

**Proof :-**

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\therefore\ \nabla.(u\,\vec{a}) = \nabla.\,(u\,a_1\hat{i} + u\,a_2\hat{j} + u\,a_3\hat{k})$

$$=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(ua_1\,\hat{i}+ua_2\hat{j}+ua_3\hat{k}\right)$$

$$=\frac{\partial}{\partial x}(ua_1)+\frac{\partial}{\partial y}(ua_2)+\frac{\partial}{\partial z}(ua_3)$$

$$=\frac{\partial u}{\partial x}a_1+u\frac{\partial a_1}{\partial x}+\frac{\partial u}{\partial y}a_2+u\frac{\partial a_2}{\partial y}_3+\frac{\partial u}{\partial z}a_3+u\frac{\partial a_3}{\partial z}$$

$$=\left(\frac{\partial u}{\partial x}a_1+\frac{\partial u}{\partial y}a_2+\frac{\partial u}{\partial z}a_3\right)+u\left(\frac{\partial a_1}{\partial x}+\frac{\partial a_2}{\partial y}+\frac{\partial a_3}{\partial z}\right)$$

$$=\left(\frac{\partial u}{\partial x}\hat{i}+\frac{\partial u}{\partial y}\hat{j}+\frac{\partial u}{\partial z}\hat{k}\right)\cdot\left(a_1\,\hat{i}+a_2\hat{j}+a_3\hat{k}\right)+u\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(a_1\,\hat{i}+a_2\hat{j}+a_3\hat{k}\right)$$

$=\nabla u\cdot\vec{a}+u\nabla.\,\vec{a}$

$=u\nabla\cdot\vec{a}+\nabla u\cdot\vec{a}$

Thus div $(u\,\vec{a})$ = udiv $\vec{a}$ + (grad u)$\cdot$ $\vec{a}$

Hence proved.

(3) Prove that div $(\vec{a}\times\vec{b}) = \vec{b}\,.$ curl $\vec{a}-\vec{a}\,.$ curl $\vec{b}$

[U.P.T.U 2003, B.P.S.C 1993).

**Proof :-** div $(\vec{a}\times\vec{b}) = \nabla\,.(\vec{a}\times\vec{b})$

$$= \sum i.\frac{\partial}{\partial x}(\vec{a}\times\vec{b})$$

$$= \sum i.\left[\frac{\partial\vec{a}}{\partial x}\times\vec{b}+\vec{a}\times\frac{\partial\vec{b}}{\partial x}\right]$$

$$= \sum \hat{i}\cdot\frac{\partial\vec{a}}{\partial x}\times b+\sum \hat{i}.\vec{a}\times\frac{\partial\vec{b}}{\partial x}$$

$$= \sum i\times\frac{\partial\vec{a}}{\partial x}.\vec{b}-\sum i\times\frac{\partial\vec{b}}{\partial x}.\vec{a}$$

$$\because \vec{a}\,.\,\vec{b}\times\vec{c} = -\,\vec{a}\times\vec{c}\,.\,\vec{b}$$

$$= \left(\sum i\times\frac{\partial\vec{a}}{\partial x}\right).\vec{b}-\left(\sum i\times\frac{\partial\vec{a}}{\partial x}\right).\vec{a}$$

$= \text{curl}\,\vec{a}\,.\,\vec{b}\ -\text{curl}\ \vec{b}\,.\,\vec{a}$

Thus div $\left(\vec{a}\times\vec{b}\right)=\vec{b}.\text{curl}\,\vec{a}-\vec{a}.\text{curl}\,\vec{b}$

Hence proved.

(4) If $\vec{a}$ is a vector function and u is a scalar function then

Curl $(u\vec{a}) = u$ curl $\vec{a}$ +(grad u)× $\vec{a}$

[U.P.T.U. (C.O) 2003]

**Proof** :- Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\therefore\ \nabla.(u\vec{a}) = \nabla.\left(u a_1\hat{i} + u a_2\hat{j} + u a_3\hat{k}\right)$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\left(ua_1\,\hat{i}+ua_2\hat{j}+ua_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ ua_1 & ua_2 & ua_3 \end{vmatrix}$$

$$= \sum \hat{i}\left\{\frac{\partial}{\partial y}(ua_3) - \frac{\partial}{\partial z}(ua_2)\right\}$$

$$= \sum \hat{i}\left[u\frac{\partial a_3}{\partial y} + \frac{\partial u}{\partial y}a_3 - u\frac{\partial a_2}{\partial z} - \frac{\partial u}{\partial z}a_2\right]$$

$$= u\sum \hat{i}\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) + \sum \hat{i}\left(\frac{\partial u}{\partial y}a_3 - \frac{\partial u}{\partial z}a_2\right)$$

$$= u\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$= u\,(\nabla \times \vec{a}) + (\nabla u) \times \vec{a}$

Thus curl $(u\ \vec{a})$= u curl $\vec{a}$ +(grad u) × $\vec{a}$

Hence proved.

(5) Prove that $\text{Curl}\left(\vec{a}\times\vec{b}\right) = \left(\vec{b}.\nabla\right)\vec{a} - \left(\vec{a}.\nabla\right)\vec{b} + \vec{a}\ \text{div}\ \vec{b} - \vec{b}\,\text{div}\,\vec{a}$

(IA.S 2000, U.P.P.C.S 1996)

**Proof :-** Curl $\left(\vec{a}\times\vec{b}\right) = \hat{i}\times\frac{\partial}{\partial x}\left(\vec{a}\times\vec{b}\right) + \hat{j}\times\frac{\partial}{\partial y}\left(\vec{a}\times\vec{b}\right) + \hat{k}\times\frac{\partial}{\partial z}\left(\vec{a}\times\vec{b}\right)$

$$= \sum \hat{i}\times\left[\frac{\partial \vec{a}}{\partial x}\times\vec{b} + \vec{a}\times\frac{\partial \vec{b}}{\partial x}\right]$$

$$= \sum \hat{i}\times\left(\frac{\partial \vec{a}}{\partial x}\times\vec{b}\right) + \sum \hat{i}\times\left(\vec{a}\times\frac{\partial \vec{b}}{\partial x}\right)$$

$$= \sum\left[(\hat{i}.\vec{b})\frac{\partial \vec{a}}{\partial x} - \left(\hat{i}.\frac{\partial \vec{a}}{\partial x}\right)\vec{b} + \left(\hat{i}.\frac{\partial \vec{b}}{\partial x}\right)\vec{a} - (\hat{i}.\vec{a})\frac{\partial \vec{b}}{\partial x}\right]$$

$\because \vec{a}\times\left(\vec{b}\times\vec{c}\right) = \left(\vec{a}.\vec{c}\right)\vec{b} - \left(\vec{a}.\vec{b}\right)\vec{c}$

$$= \left(\sum \vec{b}.\hat{i}\frac{\partial}{\partial x}\right)\vec{a} - \vec{b}\sum \hat{i}.\frac{\partial \vec{a}}{\partial x} + \vec{a}\sum \hat{i}.\frac{\partial \vec{b}}{\partial x} - \left(\sum \vec{a}.\hat{i}\frac{\partial}{\partial x}\right)\vec{b}$$

$= \ (\vec{b}\,.\nabla)\,\vec{a} - \vec{b}\,\text{div}\,\vec{a} + \vec{a}\,\text{div}\,\vec{b} - (\vec{a}\,.\nabla)\,\vec{b}$

$= (\vec{b}\,.\nabla)\,\vec{a} - (\vec{a}\,.\nabla)\,\vec{b} + \vec{a}\,\text{div}\,\vec{b} - \vec{b}\,\text{div}\,\vec{a}$

Thus curl $(\vec{a}\times\vec{b}) = (\vec{b}\,.\nabla)\,\vec{a} - (\vec{a}\,.\nabla)\,\vec{b} + \vec{a}\,\text{div}\,\vec{b} - \vec{b}\,\text{div}\,\vec{a}$

Hence proved.

## SECOND ORDER DIFFERENTIAL OPERATORS, LAPLACE'S OPERATOR $\nabla^2$ :-

The operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the Laplace's operator and the equation $\nabla^2\phi = 0$ is called the Laplace's equation. Now we shall prove some results of second order differential operators.

(1) div grad $\phi = \nabla.\nabla\,\phi = \nabla^2\phi$

**Proof :-** We have

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$\because \nabla\cdot\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$= \nabla^2\phi$$

Thus div grad $\phi = \nabla^2\phi$, hence proved.

(2) Curl grad $\phi = \nabla\times(\nabla\phi) = \vec{0}$

**Proof :-** Curl grad $\phi = \nabla\times(\nabla\phi)$

$$= \nabla\times\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) + \hat{j}\left(\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z}\right) + \hat{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right)$$

$$= \vec{0}$$

Thus curl grad $\phi = \vec{0}$. Hence proved

(3) div (curl $\vec{V}$) = 0

**Proof :-** Let $\vec{V} = V_1\hat{a} + V_2\hat{j} + V_3\hat{k}$, we have

$$\text{Curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$\therefore$ div (Curl $\vec{V}$) = $\nabla . (\nabla \times \vec{V})$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) . \left[\hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$$= \left(\frac{\partial^2 V_3}{\partial x \partial y} - \frac{\partial^2 V_2}{\partial x \partial z}\right) + \left(\frac{\partial^2 V_1}{\partial y \partial z} - \frac{\partial^2 V_3}{\partial y \partial x}\right) + \left(\frac{\partial^2 V_2}{\partial z \partial x} - \frac{\partial^2 V_1}{\partial z \partial y}\right)$$

$= 0$

Thus div (curl $\vec{V}$) = 0

Hence proved.

(4) Prove that grad div $\vec{V}$ = Curl Curl $\vec{V} + \nabla^2 \vec{V}$

OR

Curl (Curl $\vec{V}$) = grad (div $\vec{V}$) - $\nabla^2 \vec{V}$

(I.A.S 2002, U.P.P.C.S. 2003, U.P.T.U. Special Exam 2001, U.P.T.U 2003)

**Proof** :- Let $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

Then

$$\text{Curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$\therefore$ Curl Curl $\vec{V} = \nabla \times (\nabla \times \vec{V})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \left(\dfrac{\partial a_3}{\partial y} - \dfrac{\partial a_2}{\partial z}\right) & \left(\dfrac{\partial a_1}{\partial z} - \dfrac{\partial a_3}{\partial x}\right) & \left(\dfrac{\partial a_2}{\partial x} - \dfrac{\partial a_1}{\partial y}\right) \end{vmatrix}$$

$$= \Sigma \hat{i} \left\{ \frac{\partial}{\partial y}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) \right\}$$

$$= \Sigma \hat{i} \left[ \frac{\partial^2 V_2}{\partial y \partial x} - \frac{\partial^2 V_1}{\partial y^2} - \frac{\partial^2 V_1}{\partial z^2} + \frac{\partial^2 V_3}{\partial z \partial x} + \frac{\partial^2 V_1}{\partial x^2} - \frac{\partial^2 V_1}{\partial x^2} \right]$$

$$= \Sigma \hat{i} \left[ \frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_2}{\partial y \partial x} + \frac{\partial^2 V_3}{\partial z \partial x} - \frac{\partial^2 V_1}{\partial x^2} - \frac{\partial^2 V_1}{\partial y^2} - \frac{\partial^2 V_1}{\partial z^2} \right]$$

$$= \Sigma \hat{i} \left[ \frac{\partial}{\partial x}\left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) V_1 \right]$$

$$= \Sigma \hat{i} \left[ \frac{\partial}{\partial x}\left(\operatorname{div} \vec{V}\right) - \nabla^2 V_1 \right]$$

$$= \hat{i} \left[ \frac{\partial}{\partial x}\left(\operatorname{div} \vec{V}\right) - \nabla^2 V_1 \right] + \hat{j} \left[ \frac{\partial}{\partial y}\left(\operatorname{div} \vec{V}\right) - \nabla^2 V_2 \right] + \hat{k} \left[ \frac{\partial}{\partial z}\left(\operatorname{div} \vec{V}\right) - \nabla^2 V_3 \right]$$

$$= \left[ \hat{i} \frac{\partial}{\partial x}\left(\operatorname{div} \vec{V}\right) + \hat{j} \frac{\partial}{\partial y}\left(\operatorname{div} \vec{V}\right) + \hat{k} \frac{\partial}{\partial z}\left(\operatorname{div} \vec{V}\right) \right] - \nabla^2 V_1 \hat{i} - \nabla^2 V_2 \hat{j} - \nabla^2 V_3 \hat{k}$$

$$= \left( \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right)\left(\operatorname{div} \vec{V}\right) - \nabla^2 \left(V_1 \hat{i} + V_2 \hat{j} + V_3 \hat{k}\right)$$

$= \text{grad div} \vec{V} - \nabla^2 \vec{V}$

or grad (div $\vec{V}$) = Curl Curl $\vec{V} + \nabla^2 \vec{V}$

Hence proved.

## ILLUSTRATIVE EXAMPLES

**Example 1 :-** If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ , Prove that

(i) div $\vec{r} = 3$ i.e, $\nabla . \vec{r} = 3$

(ii) Curl $\vec{r} = \vec{0}$ i.e, $\nabla \times \vec{r} = \vec{0}$

**Solution :-** (i) div $\vec{r} = \nabla . \vec{r}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \qquad \because \hat{i}.\hat{i} = 1,\ \hat{i}.\hat{j} = 0 \text{ etc.}$$

$$= 1+1+1$$

$$= 3$$

(ii) Curl $\vec{r} = \nabla \times \vec{r}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \hat{j}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$$= \vec{0}$$

**Example 2 :-** Prove that, for a constant vector $\vec{a}$

(i) Curl $(\vec{r} \times \vec{a}) = -2\vec{a}$ i.e, $\nabla\times(\vec{r} \times \vec{a}) = -2\vec{a}$

(ii) div $(\vec{a} \times \vec{r}) = 0$

(U.P.T.U 2002)

**Proof :-** Let us suppose that

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

and $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\therefore \vec{r} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$= \hat{i}\ (a_3y-a_2z) - \hat{j}\,(a_3x-a_1z) + \hat{k}\,(a_2x-a_1y)$

Therefore, we have

$$\text{(i) } \nabla\times\left(\vec{r}\times\vec{a}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_3y - a_2z & a_1z - a_3x & a_2x - a_1y \end{vmatrix}$$

$= \hat{i}\ (-a_1-a_1) - \hat{j}\,(a_2+a_2) + \hat{k}\ (-a_3-a_3)$

$= -2a_1\,\hat{i} - 2a_2\,\hat{j} - 2a_3\,\hat{k}$

$= -2\,(a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k})$

$= -2\,\vec{a}$

Also we have

(ii) $\Delta.\left(\vec{a}\times\vec{r}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) . \{\hat{i}\ (a_3y-a_2z) - \hat{j}\,(a_3x-a_1z) + \hat{k}\ (a_2x-a_1y)\}$

$= \frac{\partial}{\partial x}(a_3y-a_2z) - \frac{\partial}{\partial y}(a_3x-a_1z) + \frac{\partial}{\partial z}(a_2x-a_1y)$

$= 0$

**Alternative Method :-**

Since we know div $(\vec{a}\times\vec{b}) = \vec{b}\,.\,\text{Curl}\,\vec{a} - \vec{a}\ .\,\text{Curl}\,\vec{b}$

$\therefore$ div $(\vec{a}\times\vec{r}) = \vec{r}\,.\,\text{Curl}\,\vec{a} - \vec{a}\,.\,\text{Curl}\,\vec{r}$ But curl $\vec{a} = 0$, curl $\vec{r} = 0$

Hence div $(\vec{a}\times\vec{r}) = 0$

**Example 3 :-** If $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ and $r = |\vec{r}|$ show that

div grad $r^m = m\,(m+1)\,r^{m-2}$

(U.P.P.C.S 1996, U.P.T.U. 2002, 03, 04, 05)

**Solution :-**

grad $r^m = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) r^m$

$= \hat{i}\ mr^{m-1}\frac{\partial r}{\partial x} + \hat{j}\ mr^{m-1}\frac{\partial r}{\partial y} + \hat{k}\ mr^{m-1}\frac{\partial r}{\partial z}$

$= mr^{m-1}\left[\hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}\right]$

$= mr^{m-1}\left[\hat{i}\left(\frac{x}{r}\right) + \hat{j}\left(\frac{y}{r}\right) + \hat{k}\left(\frac{z}{r}\right)\right]$

$\because \vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$

$\Rightarrow r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

$\Rightarrow r^2 = x^2 + y^2 + z^2$

$\Rightarrow 2r\frac{\partial r}{\partial x} = 2x$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \text{ etc.}$$

$= mr^{m-2}(x\hat{i} + y\hat{j} + z\hat{k})$

$\therefore$ div grad $r^m = \nabla.\ [mr^{m-2}(x\hat{i} + y\hat{j} + z\hat{k})]$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).[mr^{m-2}(x\hat{i} + y\hat{j} + z\hat{k})]$$

$$= m\left[\frac{\partial}{\partial x}(xr^{m-2}) + \frac{\partial}{\partial y}(yr^{m-2}) + \frac{\partial}{\partial z}(zr^{m-2})\right]$$

$$= m\ [\{r^{m-2} + (m-2)\ xr^{m-3}\frac{\partial r}{\partial x}\} + \{r^{m-2} + (m-2)\ yr^{m-3}\frac{\partial r}{\partial y}\} + \{r^{m-2} + (m-2)zr^{m-3}\frac{\partial r}{\partial z}\}]$$

$$= 3m\ r^{m-2} + m\ (m-2)r^{m-3}\left[x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right]$$

$$= 3m\ r^{m-2} + m\ (m-2)r^{m-3}\left[x\left(\frac{x}{r}\right) + y\left(\frac{y}{r}\right) + z\left(\frac{z}{r}\right)\right]$$

$= 3m\ r^{m-2} + m\ (m-2)r^{m-4}(x^2+y^2+z^2)$

$= 3m\ r^{m-2} + m\ (m-2)r^{m-4}(r^2)$

$\because x^2+y^2+z^2 = r^2$

$= [3m + m\ (m-2)]\ r^{m-2}$

$= m\ (m+1)\ r^{m-2}$

Hence proved.

**Example 4 :** If r and $\vec{r}$ have their usual meanings, show that

(i) div $r^n\vec{r} = (n+3)r^n$

(ii) Curl $r^n\vec{r} = \vec{0}$

**Solution :-** Since $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$ so; we have

$r^n\vec{r} = r^n x\hat{i} + r^n y\hat{j} + r^n z\hat{k}$

$$\text{(i)} \therefore \text{div } r^n\vec{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(r^n x\hat{i} + r^n y\hat{j} + r^n z\hat{k})$$

$$= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

$$= r^n.1 + nr^{n-1}\frac{\partial r}{\partial x}x + r^n.1 + nr^{n-1}\frac{\partial r}{\partial y}y + r^n.1 + nr^{n-1}z\frac{\partial r}{\partial z}$$

$$= 3\,r^n + nr^{n-1}\left(x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right)$$

$$= 3\,r^n + nr^{n-1}\left(x\frac{x}{r} + y\frac{y}{r} + z\frac{z}{r}\right)$$

$$= 3r^n + nr^{n-1}\left(\frac{x^2+y^2+z^2}{r}\right)$$

$= 3r^n + nr^n$

$= (n+3)r^n$

(ii) Curl $(r^n\,\vec{r}) = \nabla\times[(r^n x)\hat{i} + (r^n y)\hat{j} + (r^n z)\hat{k}]$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \sum \hat{i}\left\{\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)\right\}$$

$$= \sum \hat{i}\left\{znr^{n-1}\frac{\partial r}{\partial y} - ynr^{n-1}\frac{\partial r}{\partial z}\right\}$$

$$= nr^{n-1}\sum \hat{i}\left\{z\left(\frac{y}{r}\right) - y\left(\frac{z}{r}\right)\right\}$$

$\because \frac{\partial r}{\partial y} = \frac{y}{r}$ etc.

$= nr^{n-2}[(zy-yz)\hat{i} + (xz-zx)\hat{j} + (xy-yx)\hat{k}]$

$= nr^{n-2}\,[0\hat{i} + 0\hat{j} + 0\hat{k}]$

$= nr^{n-2}\,[\vec{0}]$

$= \vec{0}$ Hence Proved

**Example 5:-** Prove that div $\hat{r} = \frac{2}{r}$ where $\vec{r}$ and r have their usual meanings.

**Solution :-** we have $\hat{r} = \frac{\vec{r}}{r}$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}$$

$$= \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}$$

$$\therefore \text{div } \vec{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

$$= \frac{r.1 - x\frac{\partial r}{\partial x}}{r^2} + \frac{r.1 - y\frac{\partial r}{\partial y}}{r^2} + \frac{r.1 - z\frac{\partial r}{\partial z}}{r^2}$$

$$= \frac{1}{r^2}\left[3r - \left(x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right)\right]$$

$$= \frac{1}{r^2}\left[3r - \left(x.\frac{x}{r} + y.\frac{y}{r} + z.\frac{z}{r}\right)\right]$$

$\because$ $r^2 = x^2 + y^2 + z^2$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \text{ etc.}$$

$$= \frac{1}{r^2}\left(3r^2 - \frac{r^2}{r}\right)$$

$$= \frac{1}{r^2}(2r)$$

$$= \frac{2}{r}$$

Hence proved.

**Example 6:** If $\vec{a}$ is a constant vector and $\vec{r}$ is the position vector, show that

$$\text{Curl}\left(\frac{\vec{a} \times \vec{r}}{r^3}\right) = -\frac{\vec{a}}{r^3} - \frac{3}{r^3}(\vec{a}.\vec{r})\vec{r}$$

(I.A.S. 2001)

**Solution :-** $\text{Curl}\left(\frac{\vec{a} \times \vec{r}}{r^3}\right) = \text{Curl}\left\{r^{-3}\left(\vec{a} \times \vec{r}\right)\right\}$

$= r^{-3}\,\text{curl}\left(\vec{a} \times \vec{r}\right) + \text{grad } r^{-3} \times \left(\vec{a} \times \vec{r}\right)$ ..........(i)

Since curl $(u\vec{a}) = u$ curl $\vec{a}$ + grad $u \times \vec{a}$

Also curl $\left(\vec{a} \times \vec{r}\right) = 2\vec{a}$ .............(ii) using example 2(i)

and grad $r^{-3} = -\frac{3}{r^5}\vec{r}$ .........(iii)

From (i), (ii) and (iii) we obtain

$$\text{Curl}\left(\frac{\vec{a}\times\vec{r}}{r^3}\right) = \frac{2}{r^3}\vec{a} - \frac{3}{r^5}\left\{\vec{r}\times\left(\vec{a}\times\vec{r}\right)\right\}$$

$$= \frac{2}{r^3}\vec{a} - \frac{3}{r^5}\left\{\left(\vec{r}.\vec{r}\right)\vec{a} - \left(\vec{r}.\vec{a}\right)\vec{r}\right\}$$

$$= -\frac{1}{r^3}\vec{a} + \frac{3}{r^5}\left(\vec{r}.\vec{a}\right)\vec{r}$$

Hence proved.

**Example 7 :** If $\vec{U}$ and $\vec{V}$ are irrotational vectors, then show that $\vec{U}\times\vec{V}$ is a solenoidal vector.

(I. A. S. 2004)

**Solution :-** If $\vec{U}$ and $\vec{V}$ are irrotational vectors, then by definition

$\nabla\times\vec{U} = 0$ and $\nabla\times\vec{V} = 0$ (i)

Now $\text{div}\left(\vec{U}\times\vec{V}\right) = \nabla.\left(\vec{U}\times\vec{V}\right)$

$= \vec{V}.(\nabla\times\vec{U}) - \vec{U}.(\nabla\times\vec{V})$

$= \vec{V}.\vec{0} - \vec{U}.\vec{0}$ from (i)

or $\text{div}\left(\vec{U}\times\vec{V}\right) = 0$

Hence by definition $\vec{U}\times\vec{V}$ is a solenoidal vector.

Hence proved.

**Problem 8:-** Prove that $r^n\vec{r}$ is an irrotational vector for any value of n, but is solenoidal only if n+3 = 0, where $\vec{r}$ is the position vector of a point.

(I.A.S. 2006, 2007)

**Solution :-** Curl $(r^n\vec{r}) = \vec{0}$

See example 4 (ii)

it shows that $r^n\vec{r}$ is an irrotational for any value of n.

Again div $r^n\vec{r} = (n+3)r^n$

See example 4 (i)

we get

div $r^n\vec{r} = (n+3)\,r^n$, which is zero if n+3=0 i.e. n = -3

Hence proved.

**Example 9 :-** If vector $\vec{F} = 3x\hat{i} + (x+y)\hat{j} - az\hat{k}$ is solenoidal. Find a.

**Solution :-** A vector $\vec{F}$ is said to be solenoidal, if div $\vec{F} = 0$

$\therefore \text{div } \vec{F} = \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial x}(x+y) + \frac{\partial}{\partial x}(-az)$

$= 3 + 1 - a = 0$

$\therefore$ a= 4 Answer.

**Example 10 :-** Find the constants a, b, c so that $\vec{F} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$ is irrotational.

**Solution :-** A vector $\vec{F}$ is said to be irrotational if curl $\vec{F} = 0$

$$\text{Now, Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+4z & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

or Curl $\vec{F} = (c+1)\ \hat{i} + (a-4)\ \hat{j} + (b-2)\ \hat{k}$

Now Curl $\vec{F} = 0$ if $c+1 = 0$, $a-4 = 0$ & $b-2 = 0$

$\therefore a = 4, b = 2, c = -1$ Answer.

**Example 11:-** Prove that

$\text{div}\left\{\frac{f(r)}{r}\vec{r}\right\} = \frac{1}{r^2}\frac{d}{dr}\{r^2 f(r)\}$

where $\vec{r} = r\hat{i} + y\hat{j} + z\hat{k}$

(U.P.P.C.S. 1991)

**Solution :-** $\text{div}\left\{\frac{f(r)}{r}\vec{r}\right\} = \frac{f(r)}{r}\text{ div } \vec{r} + \vec{r}\ .\ \text{grad } \frac{f(r)}{r}$

$\because \text{div } u\vec{a} = u \text{ div } \vec{a} + \vec{a}\ .\ \text{grad } u$

$= \frac{3f(r)}{r} + \vec{r}.\ \frac{r \,\text{grad}\, f(r) - f(r) \text{grad}\, r}{r^2}$ (i)

$\because \text{div } \vec{r} = 3$

Now we have

$\text{grad } f(r) = \hat{i}f'(r)\frac{\partial r}{\partial x} + \hat{j}f'(r)\frac{\partial r}{\partial y} + \hat{k}f'(r)\frac{\partial r}{\partial z}$

$= f'(r)\left[\hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}\right]$

$= f'(r)\left[\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right]$

$= \frac{f'(r)}{r}\vec{r}$ (ii)

and grad r = $\hat{i}\frac{\partial r}{\partial x}+\hat{j}\frac{\partial r}{\partial y}+\hat{k}\frac{\partial r}{\partial z}$

$=\hat{i}\frac{x}{r}+\hat{j}\frac{y}{r}+\hat{k}\frac{z}{r}$

$=\frac{\vec{r}}{r}$ (ii)

using (ii) and (iii), (i) gives

$\text{div}\left\{\frac{f(r)}{r}\vec{r}\right\}=\frac{3f(r)}{r}+\frac{rf'(r)}{r}-\frac{f(r)}{r}$

$=\frac{2f(r)}{r}+f'(r)$

$=\frac{1}{r^2}\frac{d}{dr}\{r^2 f(r)\}$

Hence proved.

**Example 12:-** Show that the vector field defined by $\vec{F}=2xyz^3\hat{i}+x^2z^3\hat{j}+3x^2yz^2\hat{k}$ is irrotational. Find also the scalar $\phi$ such that $\vec{F}$ = grad $\phi$

(I.A.S 2001, U.P.P.C.S. 2002)

**Solution :-**

$$\text{Curl } \vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2xyz^3 & x^2z^3 & 3x^2yz^2\end{vmatrix}$$

$=\hat{i}(3x^2z^2-3x^2z^2)+\hat{j}(6xyz^2-6xyz^2)+\hat{k}(2xz^3-2xz^3)$

$=\vec{0}$

Hence $\vec{F}$ is irrotational.

Now $\vec{F}=\nabla\phi$ given

or $F_1\hat{i}+F_2\hat{j}+F_3\hat{k}=\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}$

$\therefore \frac{\partial\phi}{\partial x}=F_1,\ \frac{\partial\phi}{\partial y}=F_2,\ \frac{\partial\phi}{\partial z}=F_3$

Also $d\phi=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz$

$= 2xyz^3dx + x^2z^3dy + 3x^2yz^2dz$

$= d\,(x^2yz^3)$

Integrating, we get

$\phi = x^2yz^3 + \text{constant}$

Answer.

**Example 13 :-** A fluid motion is given by $\vec{V} = (y+z)\,\hat{i} + (z+x)\,\hat{j} + (x+y)\,\hat{k}$, show that the motion is irrotational and hence find velocity potential.

(U.P.T.U. 2003, Uttarakhand T.U. 2006).

**Solution :-** We have $\vec{V} = (y+z)\,\hat{i} + (z+x)\,\hat{j} + (x+y)\,\hat{k}$

$$\text{Curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix}$$

$= (1-1)\,\hat{i} + (1-1)\,\hat{j} + (1-1)\,\hat{k}$

$= \vec{0}$

Hence $\vec{V}$ is irrotational

Now, if $\phi$ is a scalar potential then, we have

$\vec{V} = \nabla\phi$

$$\Rightarrow (y+z)\,\hat{i} + (z+x)\,\hat{j} + (x+y)\,\hat{k} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Equating the coefficients of $\hat{i}, \hat{j}, \hat{k}$ we get

$$\frac{\partial\phi}{\partial x} = y+z, \quad \frac{\partial\phi}{\partial y} = z+x \;\&\; \frac{\partial\phi}{\partial z} = x+y$$

$$\text{Also } d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$= (y+z)dx + (z+x)dy + (x+y)dz$

$= ydx + zdx + zdy + xdy + xdz + ydz$

$= ydx + xdy + zdy + ydz + xdz + zdx$

$= d(xy) + d(yz) + d(xz)$

Interating term by term we get

$\phi = xy + yz + xz + \text{constant}$ Answer

**Note :-** For an incompressible fluid

$\text{div } \vec{V} = 0$

$$\text{Now div } \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \{(y+z)\,\hat{i} + (z+x)\,\hat{j} + (x+y)\,\hat{k}\}$$

$= \frac{\partial}{\partial x}(y+z) + \frac{\partial}{\partial y}(z+x) + \frac{\partial}{\partial z}(x+y)$

$= 0+0+0$

$= 0$

Hence motion is possible for an incompressible fluid.

**Example 14 :-** For what values of b and c will

$\vec{F} = (y^2+2czx)\hat{i} + y(bx+cz)\hat{j} + (y^2+cx^2)\hat{k}$ be a gradient field?

(U.P.T.U 2006)

**Solution :-** $\vec{F}$ will be a gradient field if $\vec{F}$ is conservative vector field i.e. $\vec{F}$ is irrotational vector function. In that case curl $\vec{F} = 0$ and consequently

$\vec{F} = \text{grad } \phi$ ($\because$ curl grad $\phi = 0$ where $\phi$ is scalar potential)

$$\text{Now Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 + 2czx & ybx + ycz & y^2 + cx^2 \end{vmatrix}$$

$= (2y-yc)\hat{i} - (2cx-2cx)\hat{j} + (by-2y)\hat{k}$

$\therefore (2y-yc)\hat{i} + (by-2y)\hat{k} = 0$

which gives $2y - yc = 0$ or $c = 2$ & $by - 2y = 0$ or $b = 2$ ($\because y \neq 0$)

Hence $b = c = 2$ Answer.

**Problem 15:-** Show that the vector field $\vec{F} = \frac{\vec{r}}{r^3}$ is irrotational as well as solenoidal. Find the scalar potential.

(U.P.T.U. 2002, 05)

**Solution :-** For the vector field $\vec{F}$ to be irrotational, curl $\vec{F} = 0$

we know that curl $(u\vec{a}) = u$ curl $\vec{a}$ + (grad u) × $\vec{a}$

Therefore,

$\text{Curl}\left(\frac{1}{r^3}\vec{r}\right) = \frac{1}{r^3}\text{Curl } \vec{r} + (\text{grad } \frac{1}{r^3}) \times \vec{r}$

$= \frac{1}{r^3}(\vec{0}) + \left(-\frac{3}{r^4}\hat{r}\right) \times \vec{r}$

$\because \text{Curl } \vec{r} = \vec{0}$

$= \vec{0} - \frac{3}{r^5}(\vec{r} \times \vec{r})$

$= = \vec{0} - \vec{0} \qquad \because \vec{r} \times \vec{r} = \vec{0}$

$= \vec{0}$

Hence vector field $\vec{F}$ is irrotational.

Again, for the vector field $\vec{F}$ to be solenoidal, div $\vec{F} = 0$

we know that div $(u\vec{a}) = u \, \text{div} \, \vec{a} + \vec{a} \, . \, \text{grad} \, u$

$$\therefore \text{div}\left(\frac{\vec{r}}{r^3}\right) = \frac{1}{r^3}\text{div} \; \vec{r} + \vec{r} \, . \, \text{grad}\left(\frac{1}{r^3}\right)$$

$$= \frac{3}{r^3} + \vec{r}.\left(-\frac{3}{r^4}\frac{\vec{r}}{r}\right)$$

$\because$ div $\vec{r} = 3$

& $\vec{r} \, . \, \vec{r} = r^2$

$$= \frac{3}{r^3} - \frac{3}{r^5}r^2$$

$$= \frac{3}{r^3} - \frac{3}{r^3}$$

$$= 0$$

Hence vector field $\vec{F}$ is solenoidal.

Now, let $\vec{F} = \nabla\phi$ where $\phi$ is scalar potential

$$\Rightarrow F_1\hat{i} + F_2\hat{j} + F_3\hat{k} = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$\therefore \frac{\partial\phi}{\partial x} = F_1, \; \frac{\partial\phi}{\partial y} = F_2, \; \frac{\partial\phi}{\partial z} = F_3$$

$$\text{Also } d\phi = \frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy + \frac{\partial\phi}{\partial y}\,dz$$

$$= \frac{x}{(x^2+y^2+z^2)^{3/2}}dx + \frac{y}{(x^2+y^2+z^2)^{3/2}}dy + \frac{z}{(x^2+y^2+z^2)^{3/2}}dz$$

$$= \frac{xdx + ydy + zdz}{(x^2+y^2+z^2)^{3/2}}$$

$= d\{-\,(x^2+y^2+z^2)^{-1/2}\}$

Integrating, we get

$$\phi = -\frac{1}{\sqrt{x^2+y^2+z^2}} + \text{Constant}$$

or $\phi = -\dfrac{1}{r}$ + Constant Answer.

**Example 16 :-** Prove that a vector field $\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$ is both solenoidal and irrotational

(U.P.T.U 2009)

**Solution :-** A vector $\vec{F}$ is said to be solenoidal if div $\vec{F} = 0$

Here div $\vec{F} = \nabla . \vec{F}$

$$= \left( \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial x} + \hat{k}\frac{\partial}{\partial x} \right) \cdot \{(x^2-y^2+x)\,\hat{i} - (2xy + y)\,\hat{j}\}$$

$$= \frac{\partial}{\partial x}(x^2-y^2+x) - \frac{\partial}{\partial y}(2xy + y)$$

$= (2x + 1) - (2x + 1)$

$= 0$

$\Rightarrow$ div $\vec{F} = 0$ (i)

A vector $\vec{F}$ is said to be irrotational if Curl $\vec{F} = 0$

$$\text{Now curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y^2 + x & -(2xy + y) & 0 \end{vmatrix}$$

$= \hat{i}\,(0 - 0) + \hat{j}\,(0 - 0) + \hat{k}\,(-2y+2y)$

$= 0$

$\Rightarrow$ Curl $\vec{F} = 0$.............(ii)

Thus from (i) & (ii) the given vector is solenoidal as well as irrotational.
Hence proved.

**EXERCISE**

1. Show that $\nabla \frac{1}{r^3} = -3r^{-5}\,\vec{r}$ where

$r = |\vec{r}| = |x\hat{i} + y\hat{j} + z\hat{k}|$

2. Show that $\nabla(\vec{a}.\vec{r}) = \vec{a}$ where $\vec{a}$ is a constant vector.

3. Find a unit vector normal to the surface $xy^3z^2 = 4$ at the point (-1, -1, 2).

Ans. $\dfrac{-\hat{i} - 3\hat{j} + \hat{k}}{\sqrt{11}}$

4. Find the directional derivative of $\frac{1}{r}$ in the direction $\vec{r}$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

(U.P.T.U. 2002, 03)

Ans. $-\dfrac{1}{r^2}$

5. Find the directional derivative of $\phi = (x^2+y^2+z^2)^{-1/2}$ at the point P(3, 1, 2) in the direction of the vector $yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$

(U.P.T.U. SE 2002)

Ans. $-\frac{9}{49\sqrt{14}}$

6. If u= x+y+z, v= $x^2+y^2+z^2$, ω = xy+yz+zx, Show that grad u, grad v, grad ω are coplanar.

(U.P.T.U. 2002)

7. If θ is the acute angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2-y^2+2z = 1$ at the point (1, -2, 1), Show that

$\text{Cos}\,\theta = \frac{3}{7\sqrt{6}}$

Hint. θ be the acute angle such that θ + ϕ = π

8. Show that $\vec{a}.\nabla\left(\frac{1}{r}\right) = -\frac{\vec{a}.\vec{r}}{r^3}$

9. Evaluate $\nabla.\left[\left(\vec{a}\times\vec{r}\right)r^n\right]$ where $\vec{a}$ is a constant vector.

Answer. 0

10. Evaluate $\nabla.(r^3\vec{r})$

Answer. $6r^3$

11. If $\vec{\omega}$ is a constant vector and $\vec{V} = \vec{\omega}\times\vec{r}$, prove that div $\vec{V} = 0$.

12. If $\vec{A} = \frac{\vec{r}}{r}$, find grad div $\vec{A}$.

Answer. - $-\frac{2}{r^3}\vec{r}$

13. Prove that $\nabla.\frac{\vec{r}}{r^3} = 0$

14. Prove that div $\left[\left(\vec{r}\times\vec{a}\right)\times\vec{b}\right] = -2\vec{b}.\vec{a}$, where $\vec{a}$ and $\vec{b}$ are constant vectors.

15. Show that the vector $\vec{V} = (x+3y)\hat{i} + (y-3z)\hat{j} + (x-2z)\hat{k}$ is solenoidal.

16. If $\vec{F} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$, show that. $\nabla.\vec{F} = \frac{2}{\sqrt{x^2+y^2+z^2}}$ and $\nabla\times\vec{F} = \vec{0}$

(U.P.T.U. 2001)

17. Find Curl $\vec{V}$ where $\vec{V} = e^{xyz}\left(\hat{i}+\hat{j}+\hat{k}\right)$

Ans. $\{x(z-y)\hat{i} + y(x-z)\hat{j} + z(y-x)\hat{k}\}\, e^{xyz}$

18. Evaluate λ, μ, γ so that the vector $\vec{V}$ is given by $\vec{V} = (2x+3y+\lambda z)\,\hat{i} + (\mu x + 2y + 3z)\,\hat{j} + (2x+\gamma y+3z)\,\hat{k}$ is irrotational.

Ans. λ = 2, μ =3 γ = 3

19. Show that Curl $\left[\vec{r}\times\left(\vec{a}\times\vec{r}\right)\right]=3\vec{r}\times\vec{a}$, $\vec{a}$ being a constant vector.

20. Prove that Curl $\left[\left(\vec{r}\times\vec{a}\right)\times\vec{b}\right]=\vec{b}\times\vec{a}$, where $\vec{a}$ and $\vec{b}$ are constant vectors.

21. Evaluate curl grad $r^n$, where $r=\left|\vec{r}\right|=\left|x\hat{i}+y\hat{j}+z\hat{k}\right|$

Ans. 0

22. Find div $\vec{F}$ and Curl $\vec{F}$ where $\vec{F}$ = grad $(x^3+y^3+z^3-3xyz)$

Ans. div $\vec{F}=6(x+y+z)$

Curl $\vec{F}=\vec{0}$

23. A vector field is given by $\vec{F}=(y\sin z-\sin x)\hat{i}+(x\sin z+2yz)\hat{j}+(xy\cos z+y^2)\hat{k}$. Prove that it is irrotational and hence find its scalar potential.

Ans. $\phi = xy\sin z+\cos x+y^2z+c$

24. Show that $\nabla^2\left(\frac{x}{r^3}\right)=0$ where r is magnitude of position vector $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

(U.P.T.U. SE 2001)

25. A vector field is given by $\vec{F}=(x^2+xy^2)\hat{i}+(y^2+x^2y)\hat{j}$ show that the field is irrotational and find the scalar potential.

Ans. $\phi=\frac{x^3}{3}+\frac{y^3}{3}+\frac{x^2y^2}{2}+c$

26. (i) Show that Curl $\left(\hat{k}\times\text{grad}\frac{1}{r}\right)+\text{grad}\left(\hat{k}\,.\,\text{grad}\frac{1}{r}\right)=\vec{0}$ where r is the distance of a point (x, y, z) from the origin and $\hat{k}$ is a unit vector in the direction of OZ.

(U.P.T.U. 2001)

(ii) If f and g are two scalar point functions prove that div $(f\nabla g)=f\nabla^2 g+\nabla f.\nabla g$

(U.P.T.U. (SE) 2001)

27. Prove that div (grad $r^n$) $=\nabla^2(r^n)=n(n+1)r^{n-2}$ where $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$.

Hence show that $\nabla^2\left(\frac{1}{r}\right)=0$. Hence or otherwise evaluate $\nabla\times\left(\frac{\vec{r}}{r^2}\right)$

(U.P.T.U. 2003, 05)

Ans. $\vec{0}$

28. (i) Prove that vector $f(r)\vec{r}$ is irrotatinal.

(ii) Prove that $\nabla^2 f(r)=f''(r)+\frac{2}{r}f'(r)$

30. Prove that $\vec{b}.\nabla\left(\vec{a}.\nabla\frac{1}{r}\right)=\frac{3\left(\vec{a}.\vec{r}\right)\left(\vec{b}.\vec{r}\right)}{r^5}-\frac{\vec{a}.\vec{b}}{r^3}$

31. Prove that $\vec{a}.\left[\nabla\left(\vec{u}.\vec{a}\right)-\nabla\times\left(\vec{u}\times\vec{a}\right)\right]=\nabla.\vec{u}$, where $\vec{a}$ is a constant unit vector.

**TICK THE CORRECT ANSWER OF THE CHOICES GIVEN BELOW:-**

1. When $\phi$ (x, y, z) = $x^2y + y^2x + z^2$, then grad $\phi$ at the point (1, 1, 1) is given by

(i) $3\hat{i}+2\hat{j}+2\hat{k}$ (ii) $3\hat{i}+3\hat{j}+2\hat{k}$

(iii) $2\hat{i}+2\hat{j}+3\hat{k}$ (iv) $\hat{i}+\hat{j}+3\hat{k}$

Ans. (ii)

2. If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ then the value of $\nabla\left(\vec{a}.\vec{r}\right)$ is given by (where $\vec{a}$ is a constant vector).

(i) $\vec{0}$ (ii) $2\vec{a}$

(iii) $-2\vec{a}$ (iv) $\vec{a}$

Ans. (iv)

3. Greatest rate of increase of u = $x^2 + yz^2$ at the point (1, -1, 3) is given by

(i) 11 (ii) 9

(iii) -9 (iv) $\sqrt{11}$

Ans. (i)

4. If the directional derivative of $\phi$ = axy + byz + czx at (1,1,1) has maximum magnitude 4 in a direction parallel to x axis, For

(i) a = 2, b = -2, c = 2 (ii) a = -2, b = -2, c = 2

(iii) a = 2, b = 2, c = 2 (iv) a = -2, b = 2, c = -2

Ans. (i)

5. Unit normal to the surface $x^3+y^3 + 3xyz = 3$ at the point (1, 2, -1) is

(i) $3\hat{i}+9\hat{j}+6\hat{k}$ (ii) $3\hat{i}-9\hat{j}-6\hat{k}$

(iii) $-3\hat{i}+9\hat{j}+6\hat{k}$ (iv) $-3\hat{i}+9\hat{j}-6\hat{k}$

Ans. (iii)

6. Directional derivative of the function $\phi = 2xy + z^2$ at the point (1, -1, 3) in the direction of the vector $\hat{i}+\hat{j}+2\hat{k}$ is

(i) $\frac{14}{3}$ (ii) $\frac{9}{3}$

(iii) $\frac{3}{16}$ (iv) 3

Ans. (i)

7. Angle between the normals to the surface $xy = z^2$ at the point (4, 1, 2) and (3, 3, -3) is

(i) $90^{\circ}$ (ii) $\cos^{-1}\sqrt{\left(\frac{3}{62}\right)}$

(iii) $\cos^{-1}\sqrt{\frac{3}{7}}$ (iv) $45^0$

Ans. (ii)

8. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then div $\vec{r}$ is
(i) 0 (ii) 1
(iii) 2 (iv) 3
Ans. (iv)

9. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then Curl $\vec{r}$ is
(i) $\vec{0}$ (ii) 1
(iii) 2 (iv) 3
Ans. (i)

10. div $\hat{r}$ is given by
(i) $\frac{1}{r}$ (ii) $\frac{2}{r}$
(iii) $-\frac{1}{r^2}$ (iv) $-\frac{2}{r^{-3}}$
Ans. (ii)

11. $r^n \vec{r}$ is an irrotational vector for n is equal to
(i) 0 (ii) 3
(iii) -3 (iv) any value of n
Ans. (iv)

12. $r^n \vec{r}$ is solenoidal only if
(i) $n = 0$ (ii) $n = 3$
(iii) $n = -3$ (iv) any value of n
Ans. (iii)

13. For a solenoidal vector $\vec{F}$ the value of Curl Curl Curl Curl $\vec{F}$ is
(i) $\Delta\vec{F}$ (ii) $\nabla^2\vec{F}$
(iii) $\nabla^3\vec{F}$ (iv) $\nabla^4\vec{F}$
Ans. (iv)

14. If $\vec{F} = (x+y+1)\hat{i} + \hat{j} - (x+y)\hat{k}$ Then the value of $\vec{F}$ . curl $\vec{F}$ is
(i) 0 (ii) 1
(iii) 2 (iv) -3
Ans. (i)

15. If $A = x^2 + y^2 + z^2$ and $\vec{B} = x\hat{i} + y\hat{j} + z\hat{k}$ then div $(A\vec{B})$ is equal to
(i) 2A (ii) 3A
(iii) - 3A (iv) 5A
Ans. (iv)

16. The value of $\nabla\left|\vec{r}\right|^2$

(i) $\vec{r}$ (ii) $r^2$
(iii) $2r$ (iv) $2\vec{r}$
Ans. (iv)

17. If $\vec{a}$ and $\vec{b}$ are constant vectors then $\nabla\left(\vec{r}\,\vec{a}\,\vec{b}\right)$

(i) $\vec{a}\,.\,\vec{b}$ (ii) $\vec{a}\times\vec{b}$
(iii) $\vec{b}\times\vec{a}$ (iv) $0$
Ans. (ii)

18. The value of $f(r)\times\vec{r}$ is
(i) $\vec{0}$ (ii) $\vec{r}$
(iii) $r$ (iv) $r^2$
Ans. (i)

19. If $\phi = \log\left|\vec{r}\right|$ then $\nabla\phi$ is

(i) $\frac{1}{r^2}$ (ii) $\frac{\vec{r}}{r}$

(iii) $\frac{\vec{r}}{r^2}$ (iv) $\frac{\vec{r}}{r^3}$

(U.P.T.U. 2009)

Ans. (iii)

20. The value of $\nabla \log r^n$ is

(i) $\frac{n}{r}$ (ii) $\frac{n}{r^2}$

(iii) $\frac{n\vec{r}}{r^2}$ (iv) $\frac{n\vec{r}}{r^3}$

Ans. (iii)

21. If $\vec{V} = (x^2-y^2)\,\hat{i} + 2xy\,\hat{j} + (y^2-xy)\,\hat{k}$ then div $\vec{V}$ is
(i) $x$ (ii) $2x$
(iii) $3x$ (iv) $4x$
Ans. (iv)

22. If $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ then the value of $\nabla^2\left(\frac{1}{r}\right)$ is

(i) $0$ (ii) $3$
(iii) $2r$ (iv) $2\vec{r}$
Ans. (i)

23. If $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ is position vector, then value of $\nabla(\log r)$ is

(i) $\frac{\vec{r}}{r}$ (ii) $\frac{\vec{r}}{r^2}$

(iii) $\frac{\vec{r}}{r^3}$ (iv) None of the above

(U.P.T.U. 1999)

Ans. (ii)

24. If $\phi = x^3 + y^3 + z^3 - 3xyz$, then the value of div grad $\phi$ is

(i) $x+y+z$ (ii) $6(x+y+z)$

(iii) $9(x+y+z)$ (iv) 0

Ans. (ii)

25. If $\phi = x^3 + y^3 + z^3 - 3xyz$, then Curl grad $\phi$ is

(i) $x+y+z$ (ii) $6(x+y+z)$

(iii) $9(x+y+z)$ (iv) 0

Ans. (iv)

26. The value of div (grad $\frac{1}{r}$) is

(i) $\frac{1}{r}$ (ii) $\frac{1}{r^2}$

(iii) $\frac{\vec{r}}{r^2}$ (iv) 0

Ans. (iv)

27. The magnitude of the vector drawn in a direction perpendicular to the surface $x^2+2y^2+z^2 = 7$ at the point (1, -1, 2) is

(i) $\frac{2}{3}$ (ii) $\frac{3}{2}$

(iii) 3 (iv) 6

Ans. (iv)

28. A unit normal to the surface $z = 2xy$ at the point (2, 1, 4) is

(i) $2\hat{i} + 4\hat{j} - \hat{k}$ (ii) $2\hat{i} + 4\hat{j} + \hat{k}$

(iii) $\frac{1}{\sqrt{21}}(2\hat{i} + 4\hat{j} - \hat{k})$ (iv) $\frac{1}{\sqrt{21}}(4\hat{i} + 2\hat{j} + \hat{k})$

Ans. (iii)

29. If $\vec{a}$ being a constant vector then the value of Curl$\left[\vec{r} \times (\vec{a} \times \vec{r})\right]$ is

(i) $3\vec{r} \times \vec{a}$ (ii) $-3\vec{r} \times \vec{a}$

(iii) $\vec{r} \times \vec{a}$ (iv) $\vec{a} \times \vec{r}$

Ans. (i)

30. The value of p for which the vector field $\vec{F} = (2x+y)\hat{i} + (3x-2z)\hat{j} + (x+pz)\hat{k}$ is solenoidal is

(i) 0 (ii) 2

(iii) -2 (iv) 1

31. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then $\nabla.\left(\frac{\vec{r}}{r}\right)$ is equal to

(i) 0 (ii) 3r

(iii) $r^2$ (iv) $\frac{2}{r}$

Ans. (iv)

32. If f(x,y,z)= c represent the equation of the surface. The unit normal to this surface is

(i) $\frac{\nabla f}{|\nabla f|}$ (ii) $\nabla f$

(iii) div grad f (iv)Curl grad f

Ans. (i)

33. The vector defined by $\vec{V} = e^x \sin y\,\hat{i} + e^x \cos y\,\hat{j}$ is

(i) rotational (ii) irrotational

(iii) Solenoidal (iv) Both solenoidal and irrotatinal

Ans. (iv)

34. Let $\vec{F} = x^2\hat{i} + xy\,e^x\hat{j} + \sin z\,\hat{k}$ then $\nabla.(\nabla\times\vec{F})$ equals

(i) x+cos z (ii) 0

(iii) $e^x$ (iv) $e^z + \cos z$

Ans. (ii)

**INDICATE TRUE OR FALSE FOR THE FOLLOWING STATEMENTS :-**

1. Vector having zero divergence is called solenoidal.

True/False

Ans. True

2. Vector having zero divergence is called irrotational.

True/False

Ans. False

3. The motion of the fluid having velocity $\vec{V}$ at a point P, then div $\vec{V}$ gives the rate at which fluid is originating at a point per unit volume.

True/False

Ans. True

4. A fluid motion is given by $\vec{V} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$. Is this motion irrotational?

True/False

Ans. True

5. A fluid motion is given by $\vec{V} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$, the velocity potential is equal to xy+yz+zx.

True/False

Ans. True

6. The fluid motion is given by $\vec{V} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$ the motion possible for an incompressible fluid.

True/False

Ans. True

8. Vectors having zero divergence are called solenoidal.

True/False

Ans. True

9. Vectors having zero divergence are called irrotational.

True/False

Ans. False

10. The motion of the fluid having velocity $\vec{V}$ at a point P(x, y, z). Then div $\vec{V}$ gives the rate at which fluid is originating at a point per unit volume.

True/False

Ans. True

11. If the fluid is incompressible, there can be no gain or loss in the volume element. Hence div $\vec{V}=0$. This is known in hydrodynamics as the equation of continuity for incompressible fluids.

True/False

Ans. True

12. Divergence of a constant vector $\vec{a}$ is zero.

True/False

Ans. True

13. If a rigid body is in motion, the Curl of its linear velocity at any point equal half its angular velocity.

True/False

Ans. False

14. If a rigid body is in motion, the curl of its linear velocity at any point equal twice its angular velocity.

True/False

Ans. True

15. $r^n \vec{r}$ is an irrotational for any value of n.

True/False

Ans. True

16. $r^n \vec{r}$ is solenoidal only if n = -3

True/False
Ans. True

17. If n = -1, then $\nabla^2\left(\frac{1}{r}\right)$ is zero

True/False
Ans. True

18. The vector field $\vec{F} = \frac{\vec{r}}{|\vec{r}|^3}$ is irrotational as well as solenoidal.

True/False
Ans. True

19. For a constant vector $\vec{a}$, Curl $\vec{a} = \vec{0}$

True/False
Ans. True

20. The velocity $\vec{V}$ of any point P(x, y, z) on the body is given by $\vec{V} = \vec{\omega} \times \vec{r}$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is the position vector of P. If Curl $\vec{V} = \vec{0}$ then $\vec{V}$ is said to be an irrotational vector.

True/False
Ans. True

21. The directional derivative of a scalar field $\phi$ at a point P(x, y, z) in the direction of unit vector $\vec{a}$ is given by $\hat{a}$ grad $\phi$

True/False
Ans. True

22. The temperature distribution in a medium is the example of vector point function.

True/False
Ans. False

23. The temperature distribution in a medium is the example of scalar point function.

True/False
Ans. True

24. The velocity of a moving fluid at any instant is the example of vector point function.

True/False
Ans. True

**MATCH THE ITEMS ON THE RIGHT HAND SIDE WITH THOSE ON LEFT HAND SIDE :-**

1.

| | |
|---|---|
| (i) Directional derivative | (p) $\frac{\nabla\phi}{\overline{\nabla\phi}}$ |
| (ii) Unit normal | (q) $\hat{a}\,.\text{grad}\ \phi$ |
| (iii) Greatest rate of increase | (r) grad $\phi$ |
| (iv) Directional derivative is maximum along | (s) $\lvert\nabla\phi\rvert$ |

Ans. (i, q), (ii, p), (iii, s), (iv, r)

2.

| | |
|---|---|
| (i) $\nabla.\vec{V}=0$ | (p) Solenoidal and irrotational |
| (ii) $\nabla\times\vec{V}=0$ | (q) 3 |
| (iii) $\vec{V}=\frac{\vec{r}}{r^3}$ | (r) Solenoidal |
| (iv) $\nabla.\vec{r}$ | (s) Irrotational |

Ans. (i, r), (ii, s), (iii, p), (iv, q)

3.

| | |
|---|---|
| (i) Curl $\vec{r}$ | (p) $-2\vec{a}$ |
| (ii) div $\vec{r}$ | (q) $2\vec{a}$ |
| (iii) Curl $(\vec{a}\times\vec{r})$ | (r) 0 |
| (iv) Curl $(\vec{r}\times\vec{a})$ | (s) 3 |

Ans. (i, r), (ii, s), (iii, q), (iv, p)

4.

| | |
|---|---|
| (i) $r^n\vec{r}$ is an irrotational for | (p) $n=-1$ |
| (ii) $r^n\vec{r}$ is solenoidal for | (q) $n=4$ |
| (iii) $3x\hat{i}+(x+y)\hat{j}-nz\hat{k}$ is solenoidal for | (r) $n=-3$ |
| (iv) $\nabla^2\left(\frac{1}{r}\right)=0$ for | (s) Any value of n |

Ans. (i, s), (ii, r), (iii, q), (iv, p)

5. If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

| | |
|---|---|
| (i) $\nabla r$ | (p) $-\frac{3\vec{r}}{r^5}$ |
| (ii) $\nabla\frac{1}{r}$ | (q) $\frac{2}{r}$ |

(iii) $\nabla \frac{1}{r^3}$ (r) 0

(iv) div $\frac{\vec{r}}{r}$ (s) $\hat{r}$

(v) $\nabla . \frac{\vec{r}}{r^3}$ (t) $-\frac{\hat{r}}{r^2}$

Ans. (i, s), (ii, t), (iii, p), (iv, q), (v, r)

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# Chapter 13
# Vector Integral Calculus

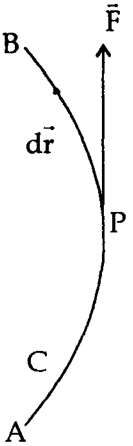
## INTEGRATION OF VECTOR FUNCTIONS :-

*LINE, SURFACE AND VOLUME INTEGRALS: LINE INTEGRALS* :- Let $\vec{r} = f(t)$ represents, a continuously differentiable curve denoted by C and f(r) be a continuous vector point function. Then $\frac{d\vec{r}}{ds}$ is a unit vector function along the tangent at and point P on the curve. The component of the vector function $\vec{F}$ along this tangent is $\vec{F}.\frac{d\vec{r}}{ds}$ which is a function of s for points on the curve. Then $\int_c \vec{F}.\frac{d\vec{r}}{ds}ds = \int_c \vec{F}.d\vec{r}$ is called the line integral or tangent line integral of $\vec{F}$ (r) along C.

Let $\vec{F} = \hat{i} F_1 + \hat{j} F_2 + \hat{k} F_3$

and $\vec{r} = \hat{i} x + \hat{j} y + \hat{k} z$

$\therefore d\vec{r} = \hat{i} dx + \hat{j} dy + \hat{k} dz$



$$\therefore \int F.dr = \int \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right).\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$
$$= \int (F_1dx + F_2dy + F_3dz)$$
$$= \int \left(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\right)dt$$

$$\therefore \int_c \vec{F}.d\vec{r} = \int_{t_1}^{t_2}\left(F_1\frac{dx}{dt}+F_2\frac{dy}{dt}+F_3\frac{dz}{dt}\right)dt$$

where $t_1$ and $t_2$ are the values of the parameter t for extermities A and B of the arc of the curve C.

Again, if $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\therefore \frac{d\vec{r}}{ds} = \frac{dx}{ds}\hat{i}+\frac{dy}{ds}\hat{j}+\frac{dz}{ds}\hat{k}$$

$$\therefore \int_c \vec{F}.d\vec{r} = \int_c \vec{F}.\frac{d\vec{r}}{ds}\,ds$$

$$= \int_{s_1}^{s_2}\left(F_1\frac{dx}{ds}+F_2\frac{dy}{ds}+F_3\frac{dy}{ds}\right)ds$$

where $s_1$ and $s_2$ are the values of s for the extermities of A and B of the arc C.

**Illustrative Examples**

**Example :-** If $\vec{F} = 3xy\,\hat{i} - y^2\,\hat{j}$ evaluate $\int_c \vec{F}.d\vec{r}$, where C is the curve in the xy plane $y= 2x^2$ from (0, 0) to (1, 2).

(B.P.S.C 2002).

**Solution :-** In the xy plane z = 0, hence $d\,\vec{r} = dx\,\hat{i} + dy\,\hat{j}$

$$\therefore \int_c \vec{F}.d\vec{r} = \int_c \left(3xy\hat{i} - y^2\hat{j}\right).\left(dx\hat{i}+dy\hat{j}\right)$$

$$= \int_c 3xy\,dx - \int_c y^2dy$$

put $y = 2x^2$ $\therefore dy = 4x\,dx$ and x varies from 0 to 1.

$$\therefore \int_c F.dr = \int_0^1 3x(2x^2)dx - \int_0^1 (2x^2)^2.4xdx$$

$$= \int_0^1 6x^3\,dx - \int_0^1 16x^5\,dx$$

$$= \left[\frac{6}{4}x^4 - \frac{16}{6}x^6\right]_0^1$$

$$= -\frac{7}{6}\text{ Answer.}$$

**Example 2:-** Evaluate $\int_c \vec{F}.d\vec{r}$, where $\vec{F} = xy\,\hat{i} + yz\,\hat{j} + zx\,\hat{k}$ and where C is $\vec{r} = \hat{i}\,t + \hat{j}\,t^2 + \hat{k}\,t^3$, t varying from -1 to +1.

**Solution:-** The equation of the curve in parametric form is

$x = t, y = t^2, z = t^3$

$\therefore \vec{F} = xy\,\hat{i} + yz\,\hat{j} + zx\,\hat{k}$

$\Rightarrow \vec{F} = t^3\,\hat{i} + t^5\,\hat{j} + t^4\,\hat{k}$

Also $\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$

$= \hat{i} + 2t\hat{j} + 3t^2\hat{k}$

$\therefore \vec{F}.\frac{d\vec{r}}{dt} = t^3 + 2t^6 + 3t^6$

$= t^3 + 5t^6$

$$\therefore \int_c \vec{F}.d\vec{r} = \int_c \vec{F}.\frac{d\vec{r}}{dt}dt$$

$$= \int_{-1}^{1}\left(t^3 + 5t^6\right)dt$$

$$= \left[\frac{t^4}{4} + \frac{5t^7}{7}\right]_{-1}^{1}$$

$$= \frac{10}{7} \text{ Answer.}$$

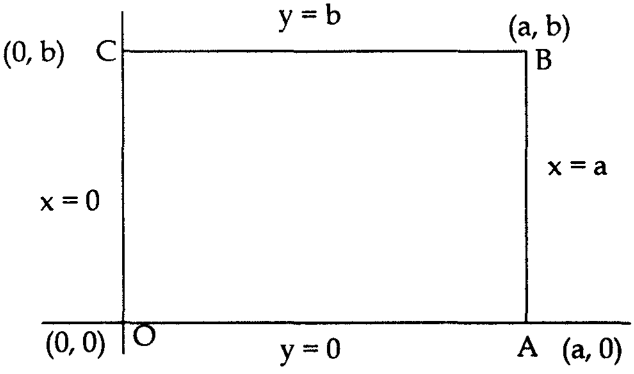
**Example 3:-** Evaluate $\int_c \vec{F}.d\vec{r}$, where $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ and the curve C is the rectangle in the xy plane bounded by y = 0, x = a, y = b, x = 0

(U.P.T.U. 2002, U.P.P.C.S 1997)

**Solution:-** In the xy plane, z=0

$\therefore \vec{r} = x\hat{i} + y\hat{j}$

$\Rightarrow d\vec{r} = dx\hat{i} + dy\hat{j}$



$$\therefore \int_c \vec{F}.d\vec{r} = \int_c \left\{\left(x^2 + y^2\right)dx - 2xy\,dy\right\} \quad \text{(i)}$$

$$\text{Now } \int_c \vec{F}.d\vec{r} = \int_{OA} \vec{F}.d\vec{r} + \int_{AB} \vec{F}.d\vec{r} + \int_{BC} \vec{F}.d\vec{r} + \int_{CO} \vec{F}.d\vec{r} \quad \text{(ii)}$$

Along OA, y = 0

$\therefore$ dy = 0 and x varies from 0 to a

Along AB , x = a

$\therefore$ dx = 0 and y varies from 0 to b

Along BC, y = b

$\therefore$ dy = 0 and x varies from a to 0

Along CO, x = 0

$\therefore$ dx = 0 and y varies from b to 0

Hence from (i) and (ii) we get

$$\int_c \vec{F}.d\vec{r} = \int_0^a x^2\,dx - \int_0^b 2ay\,dy + \int_a^o (x^2+b^2)dx + \int_b^o o\,dy$$

$$= \frac{a^3}{3} - 2a\frac{b^2}{2} + \left(\frac{x^3}{3} + b^2x\right)_a^o + O$$

$$= \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - b^2a$$

= - 2ab$^2$ Answer

**Example 4:-** Compute the work done by the force $\vec{F} = (2y+3)\hat{i} + xy\hat{j} + (yz-x)\hat{k}$ when it moves a particle from the point (0, 0, 0) to the point (2, 1, 3) along the curve $x = 2t^2$, $y = t$, $z = t^5$

**Solution:-**

$x = 2t^2$, $y = t$, $z = t^3$

$\therefore dx = 4\,t\,dt$, $dy = dt$, $dz = 3t^2\,dt$

At point (0, 0, 0),

x= 0, t = 0, y = 0, t = 0, z = 0, t = 0

At point (2, 1, 1)

x= 2, t = 1, y = 1, t = 1, z = 1, t = 1

$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$= 4t\,dt\hat{i} + dt\hat{j} + 3t^2\,dt\hat{k}$

$\vec{F} = (2t+3)\ \hat{i}\ + 2t^5\hat{j} + (t^4-2t^2)\ \hat{k}$

$\vec{F}.d\vec{r} = [(2t+3)\ \hat{i} + 2t^5\hat{j} + (t^4-2t^2)\ \hat{k}].\ [4t\,dt\hat{i} + dt\hat{j} + 3t^2\,dt\hat{k}]$

$= (8t^2+12t)\,dt + 2t^5\,dt + (3t^6-6t^4)\,dt$

work done = $\int_c \vec{F}.d\vec{r}$

$$= \int_0^1 (8t^2+12t)dt + \int_0^1 2t^5\,dt + \int_0^1 (3t^6-6t^4)dt$$

$$= \left[\frac{8t^3}{3} + \frac{12t^2}{2}\right]_0^1 + \left[\frac{2t^6}{6}\right]_0^1 + \left[\frac{3t^7}{7} - \frac{6t^5}{5}\right]_0^1$$

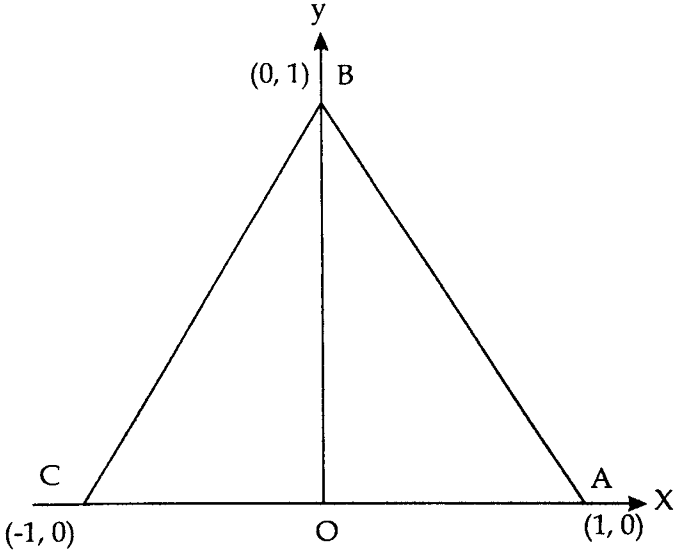
$$= \left(\frac{8}{3} + 6\right) + \frac{1}{3} + \left(\frac{3}{7} - \frac{6}{5}\right)$$

$= 8\frac{8}{35}$ Answer.

**Example 5:-** Evaluate the line integral $\int_c (y^2\,dx - x^2\,dy)$ about the triangle whose vertices are (1, 0), (0,1) and (-1, 0).

(P.T.U 1999)

**Solution:-** Let A (1, 0), B(0, 1) and C (-1, 0) be the points of curves as shown in figure.



Line integral along AB :

Equation of AB is $\frac{x}{1}+\frac{y}{1}=1$

or x+y = 1

$\Rightarrow$ dx + dy =0

Consider $\int_{AB}(y^2\,dx - x^2\,dy) = \int_{AB}\{(1-x)^2\,dx - x^2(-dx)\}$

$= \int_{AB}(1+x^2-2x+x^2)\,dx$

$= \int_{AB}(1+2x^2-2x)\,dx$

$= \int_0^1(1+2x^2-2x)\,dx$

$= \left[x+\frac{2x^3}{3}-\frac{2x^2}{2}\right]_0^1 = -\frac{2}{3}$

**Example 6:-** Evaluate $\iint\limits_S \vec{F}.\hat{n}\,ds$, where $\vec{F} = yz\,\hat{i} + zx\,\hat{j} + yx\,\hat{k}$ and S is that of the surface of the sphere $x^2+y^2+z^2= a^2$ which lies in the first octant.

(U.P.T.U. 2005, MDU 2002)

**Solution:-** The given equation of the sphere is $\phi \equiv x^2+y^2+z^2-a^2 = 0$

grad $\phi = \Delta\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\hat{n}$ = Unit normal in the direction of grad $\phi$

or $\hat{n} = x\hat{i} + y\hat{j} + z\hat{k}$

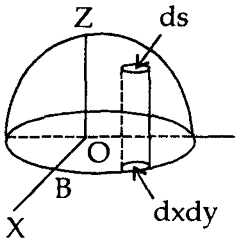
Also $\overline{dS} = \hat{n}.dS$

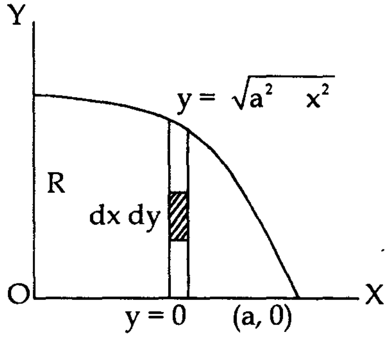
Let $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$

$\therefore \hat{n}.\hat{k} = z$

and $\vec{F}.\hat{n} = (yz\hat{i} + zx\hat{j} + xy\hat{k}).(x\hat{i} + y\hat{j} + z\hat{k})$

$= 3xyz$





**Now** $\int_S (yz\hat{i} + zx\hat{j} + xy\hat{k}).\vec{dS} = \iint_R \vec{F}.\hat{n}\frac{dxdy}{|\hat{n}.\hat{k}|}$, where R is the projection of the surface S.

$$= \iint_R 3xyz\frac{dxdy}{z}$$

$$= 3\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} xy\,dy\,dx$$

$$= 3\int_{x=0}^{a} x\left[\frac{y2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{3}{2}\int_0^a x(a^2-x^2)dx$$

$$= \frac{3}{2}\left(\frac{a^2x^2}{2} - \frac{x^4}{4}\right)_0^a = \frac{3a^4}{8} \text{ Answer}$$

**Green's Theorem in Cartesian form :-**

If C be a regular closed curve in the xy plane and S be the region bounded by C then

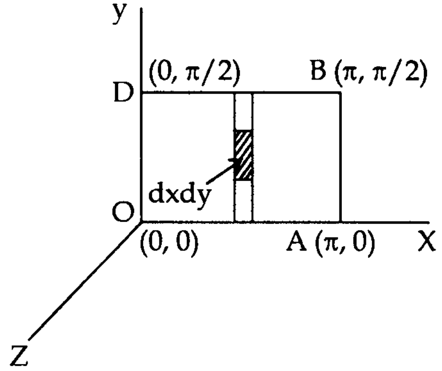
$\int_C (Pdx+Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$ where P and Q are Continuously differential functions inside and on C.

**Example 1:-** Evaluate $\int_C (e^{-x} \sin y\,dx + e^{-x} \cos y\,dy)$ by Green's theorem where C is the rectangle whose vertices are (0, 0), (π, 0), $(\pi, \frac{\pi}{2})$, and (0, π/2).

(I.A.S 1999, U.P.T.U. SC 2006 - 07 )

**Solution :-** Comparing the given line integral with the integral on the left of $\int_C (Pdx+Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$ we have $P = e^{-x} \sin y$, $Q = e^{-x} \cos y$

$\therefore \frac{\partial P}{\partial y} = e^{-x} \cos y$ & $\frac{\partial Q}{\partial x} = -e^{-x} \cos y$



Hence by Green's theorem (Cartesian form) we have the given integral

$= \int_S (-e^{-x} \cos y - e^{-x} \cos y)\,dx\,dy$

$= -2 \int_{x=0}^{\pi} \int_{y=0}^{\pi/2} e^{-x} \cos y\,dx\,dy$

$= -2\left[-e^{-x}\right]_0^{\pi} (\sin y)_0^{\pi/2}$

$= 2\,(e^{-\pi}-1)\,(1)$

$=2(e^{-\pi}-1)$ Answer

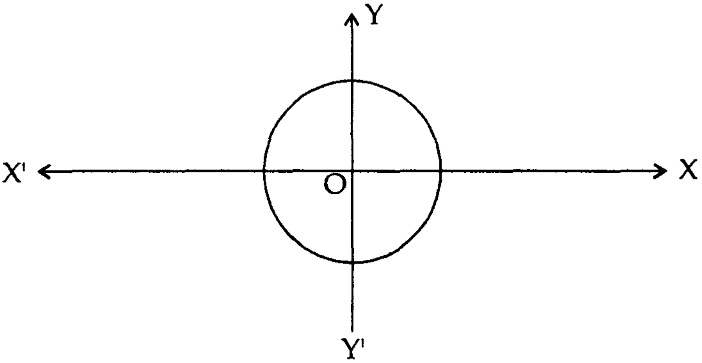
**Example 2:-** Apply Green's theorem to evaluate $\int_C (2x^2 - y^2)\,dx + (x^2 + y^2)\,dy$ where C is the boundary of the area enclosed by the x axis and the upper half of circle $x^2+y^2 = a^2$.

(U.P.T.U. 2005)

**Solution** :- Comparing the given integral with the integral on the left of $\int_C (Pdx+Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$ we have

$P = 2x^2 - y^2,\ Q = x^2 + y^2$

$\therefore \frac{\partial P}{\partial y} = -2y\,,\ \frac{\partial Q}{\partial x} = 2x$



Hence by Green's theorem, we have the given integral

$$= \iint_S (2x + 2y)\,dx\,dy$$

$$= 2\int_{x=-a}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} (x + y)\,dx\,dy$$

$$= 2\int_{-a}^{a} \left(xy + \frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= 2\int_{-a}^{a} \left(x\sqrt{a^2 - x^2} + \frac{a^2 - x^2}{2}\right) dx$$

$$= 2\int_{-a}^{a} x\sqrt{a^2 - x^2}\,dx + \int_{-a}^{a} (a^2 - x^2)\,dx$$

$$= 0 + 2\int_0^a \left(a^2 - x^2\right)dx$$

$$= 2\left(a^2x - \frac{x^3}{3}\right)_0^a$$

$$= 2\left(a^3 - \frac{a^3}{3}\right)$$

$$= 2\left(\frac{2a^3}{3}\right) = \frac{4}{3}a^3 \text{ Answer}$$

**Example 3:-** Using Green's theorem, evaluate $\int_C (x^2y\,dx + x^2\,dy)$ where C is the boundary described counter clockwise of the triangle with vertices (0, 0), (1, 0), (1,1).

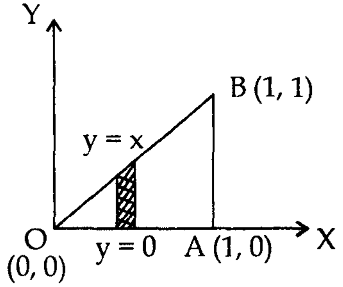
(U.P.T.U. 2004)

**Solution :-** Comparing the given integral with the integral on the left of

$$\int_C (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

Hence $P = x^2y$, $Q = x^2$

$$\therefore \frac{\partial P}{\partial y} = x^2, \frac{\partial Q}{\partial x} = 2x$$



Hence by Green's theorem, we have the given integral

$$= \iint_S (2x - x^2)dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{x}(2x - x^2)dx\,dy$$

$$= \int_0^1 (2x - x^2)\,(y)_0^x\,dx$$

$= \int_0^1 (2x^2 - x^3)dx$

$= \left(\frac{2}{3}x^3 - \frac{x^4}{4}\right)_0^1$

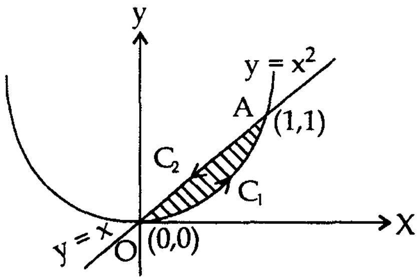
$= \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$ Answer.

**Example 4:-** Verify Green's theorem in the plane for $\int_C (xy + y^2)dx + x^2dy$ where C is the closed curve of the region bounded by y = x and $y = x^2$

(P.T.U. 2000, 05, V.T.U. 2004)

**Solution :-** Here $P = xy+y^2$, $Q = x^2$

$\therefore \frac{\partial P}{\partial y} = x + 2y$, $\frac{\partial Q}{\partial x} = 2x$



Let R be the region bounded by C,
Along $C_1$, $y = x^2$
$\therefore dy = 2xdx$ and the limit of x are 0 to 1.
$\therefore$ Line integral along $C_1 = \int_{C_1} (Pdx + Qdy)$

$= \int_0^1 (xy + y^2)dx + x^2\,dy$

$= \int_0^1 \left[\{x.x^2 + (x^2)^2\}dx + x^2.2x\,dx\right]$

$= \int_0^1 \{(x^3 + x^4)dx + 2x^3\,dx\}$

$= \int_0^1 (3x^3 + x^4)dx$

$= \left[\frac{3}{4}x^4 + \frac{x^5}{5}\right]_0^1 = \left(\frac{3}{4} + \frac{1}{5}\right) = \frac{19}{20}$

Along $C_2$, y=x from point A to O , dy = dx, and the limits of y are 1 to 0.
Line integral along

$$C_2 = \int_{C_2} (P\,dx + Q\,dy)$$

$$= \int_1^0 \left\{(xy + y^2)dx + x^2 dy\right\}$$

$$= \int_1^0 \left\{(y^2 + y^2)dy + y^2\,dy\right\}$$

$$= \int_1^0 (2y^2 + y^2)dy$$

$$= \int_1^0 3y^2 dy = \left[y^3\right]_1^0 = -1$$

Therefore, line integral along C

$$= \frac{19}{20} - 1 = -\frac{1}{20}$$

i.e. $\int_C (P\,dx + Q\,dy) = -\frac{1}{20}$ (i)

Now

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_R \left[2x - (x + 2y)\right] dx\,dy$$

$$= \int_0^1 \int_{x^2}^{x} (x - 2y)dy\,dx$$

$$= \int_0^1 \left(xy - y^2\right)_{x^2}^{x} dx$$

$$= \int_0^1 (x^2 - x^2 - x^3 + x^4)dx$$

$$= \int_0^1 (x^4 - x^3)dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \left(\frac{1}{5} - \frac{1}{4}\right) = -\frac{1}{20} \qquad \text{(ii)}$$

From (i) & (ii)

$$\int_C (P\,dx + Q\,dy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Hence Green's theorem is verified.

**Example 5:-** Evaluate $\int_C (y - \sin x)dx + \cos x\,dy$, where C is the triangle formed by $y = 0$, $x = \frac{\pi}{2}$, $y = \frac{2}{\pi}x$, by using Green's theorem.

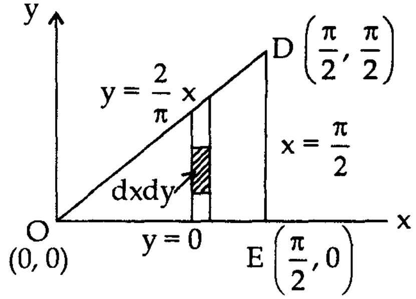
**Solution:-** The vertices of the triangle OED are (0, 0), $(\frac{\pi}{2}, 0)$ and $(\frac{\pi}{2}, \frac{\pi}{2})$

Now by Green's theorem, we have

$$\int_C (Pdx+Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Here P = y-sin x , Q = cos x

$$\therefore \frac{\partial P}{\partial y} = 1, \ \frac{\partial Q}{\partial x} = -\sin x$$


y
x
$D\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \frac{2}{\pi}x$
$x = \frac{\pi}{2}$
dxdy
O
(0, 0)
y = 0
$E\left(\frac{\pi}{2}, 0\right)$

$$\therefore \text{R.H.S} = \int_0^{\pi/2} \int_{y=0}^{\frac{2x}{\pi}} (-\sin x - 1)dx\,dy$$

$$= -\int_0^{\pi/2} (1+\sin x)\left[y\right]_0^{2x/\pi}$$

$$= -\frac{2}{\pi}\int_0^{\pi/2} (x + x\sin x)dx$$

$$= -\frac{2}{\pi}\left[\frac{x^2}{2} + x(-\cos x) - 1(-\sin x)\right]_0^{\pi/2}$$

$$= -\frac{2}{\pi}\left[\frac{1}{2}.\frac{\pi^2}{4} + 1\right]$$

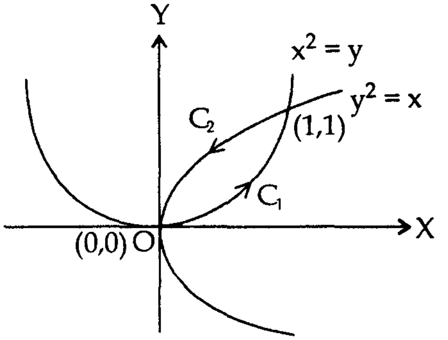
$$= -\left[\frac{\pi}{4} + \frac{2}{\pi}\right] \text{Answer}$$

**Example 6:-** Verify Green's theorem in the plane $\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of the region defined by $y = \sqrt{x}$, $y = x^2$.

**Solution:-** $y = \sqrt{x}$ i.e $y^2 = x$ and $y = x^2$ are two parabolas which intersect at (0, 0) and (1, 1)

we have

$$\int_C (P\,dx + Q\,dy) = \int_{C_1} (P\,dx + Q\,dy) + \int_{C_2} (P\,dx + Q\,dy)$$



Along $C_1$; $x^2 = y$; $\therefore dy = 2x\,dx$ and limits of x are from 0 to 1.

$\therefore$ Line integral along $C_1$ becomes $\int_0^1 \left\{\left(3x^2 - 8x^4\right)dx + \left(4x^2 - 6x^3\right)2x\,dx\right\}$

$$= \int_0^1 \left(3x^2 + 8x^3 - 20x^4\right)dx$$

$= 1 + 2 - 4 = -1$

Along $C_2$, $y^2 = x$, $\therefore 2y\,dy = dx$ and limits of y are from 1 to 0.

$\therefore$ Line integral along $C_2$ becomes

$$\int_1^0 \left(3y^4 - 8y^2\right)2y\,dy + \left(4y - 6y^2 \cdot y\right)dy$$

$$= \int_1^0 \left(6y^5 - 22y^3 + 4y\right)dy$$

$$= -\left[6.\frac{1}{6} - 22.\frac{1}{4} + 4.\frac{1}{2}\right] = \frac{5}{2}$$

$\therefore$ Line integral along C $= -1 + \dfrac{5}{2} = \dfrac{3}{2}$ (i)

Again $\int_C (P\,dx + Q\,dy) = \iint_S \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)dx\,dy$

$$=\iint_S\left[\frac{\partial}{\partial x}(4y-6xy)-\frac{\partial}{\partial y}(3x^2-8y^2)\right]dx\,dy$$

$$=\iint_S(-6y+16y)\,dx\,dy$$

$$=\int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}}10y\,dx\,dy$$

$$=\int_{x=0}^{1}\left[5y^2\right]_{x^2}^{\sqrt{x}}dx$$

$$=5\int_0^1(x-x^4)\,dx=5\left(\frac{1}{2}-\frac{1}{5}\right)$$

$$=5.\frac{3}{10}$$

$$=\frac{3}{2} \qquad \text{(ii)}$$

Hence from (i) and (ii) the Green's theorem is verified.

**Example 7:-** Apply Green's theorem to prove that the area enclosed by a plane curve is $\frac{1}{2}\int_C x\,dy-y\,dx$. Hence find the area of an ellipse whose semi- major and minor axes are of lengths a and b.

(V.T.U. 2000).

**Solution:-** By Green's theorem $\oint_C(Pdx+Qdy)=\iint_S\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$

Let $P=-\frac{y}{2}$ and $Q=\frac{x}{2}$

$$\therefore\oint_C\left(-\frac{y}{2}dx+\frac{x}{2}dy\right)=\iint_S\left\{\frac{\partial}{\partial x}\left(\frac{x}{2}\right)-\frac{\partial}{\partial y}\left(-\frac{y}{2}\right)\right\}dx\,dy$$

$$=\iint_S\left(\frac{1}{2}+\frac{1}{2}\right)dx\,dy$$

$$=\iint_S dx\,dy$$

= Area of a closed curve

∴ Area of a closed curve

$$=\frac{1}{2}\oint_C(x\,dy-y\,dx)$$

Let the equation of ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Area of ellipse $= \frac{1}{2}\oint_C (x\,dy - y\,dx)$

where C is ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

i.e. x= a cos t, y = b sin t and t varies from 0 to 2π.

∴ Required area of ellipse

$= \frac{1}{2}\int_0^{2\pi} ab(\cos^2 t + \sin^2 t)dt$

$= \frac{ab}{2}[t]_0^{2\pi}$

$\because \frac{dx}{dt} = -a \sin t,\ \frac{dy}{dt} = b \cos t$

$= \pi ab$

**Gauss Divergence Theorem :**

**Statement :-** If V is the volume bounded by a closed surface S and $\vec{F}$ is a vector point function with continuous derivatives, then

$\int_S \vec{F}.\hat{n}\,ds = \int_V \text{div}\vec{F}\,dV$

where $\hat{n}$ is unit outward drawn normal vector to the surface S.

**Example 1:** If V is the volume enclosed by a closed surface S and $\vec{F} = x\hat{i} + 2y\hat{j} + 3z\hat{k}$ show that $\int_S \vec{F}.\hat{n}\,ds = 6V$

**Solution :-** Given $\vec{F} = x\hat{i} + 2y\hat{j} + 3z\hat{k}$

∴ div $\vec{F} = \nabla.\vec{F}$

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(x\hat{i} + 2y\hat{j} + 3z\hat{k}\right)$

$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(3z)$

$= 1+2+3 = 6$

∴ By Gauss theorem we have

$\int_S \vec{F}.\hat{n}\,ds = \int_V \text{div}\vec{F}\,dv$

$= \int_V 6\,dV$

= 6V Hence proved.

**Example 2:-** Using Gauss divergence theorem, Prove that

(i) $\int \vec{r}.\hat{n}\,ds = 3V$ (ii) $\int\limits_S \nabla r^2 .ds = 6V$

where S is any closed surface enclosing a volume V and $r^2 = x^2+y^2+z^2$

(U.P.T.U. 2003)

**Solution :-** (i) By Gauss divergence theorem, we have

$$\iint\limits_S \vec{r}.\hat{n}\,ds = \iiint\limits_V \operatorname{div}\vec{r}\,dV$$

where V is the volume enclosed by the surface S.

$$= \iiint\limits_V 3dv \quad \because \text{ we know div } \vec{r} = 3$$

$= 3V$ Answer

(ii) By divergence theorem

$$\int\limits_S \nabla r^2\,ds = \int\limits_V \nabla.(\nabla r^2)dv$$

$$= \int\limits_V \nabla^2 r^2\,dv$$

$\because \nabla.\nabla = \nabla^2$

$$= \int\limits_V 2(2+1)r^{2-2}\,dv$$

$\because \nabla^2 r^2 = n(n+1)\, r^{n-2}$

$$= 6\int dv$$

$= 6V$ Answer

**Example 3:-** Show that $\int\limits_S (ax\hat{i} + by\hat{j} + cz\hat{k}).\hat{n}\,ds = \frac{4}{3}\pi\,(a+b+c)$

where S is the surface of the sphere $x^2+y^2+z^2 = 1$

**Solution :-** We have by Gauss's divergence theorem.

$$\int\limits_S \vec{F}.\hat{n}\,ds = \int\limits_V \operatorname{div}\vec{F}\,dv$$

Now

$\operatorname{div}\vec{F} = \nabla.\vec{F}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(ax\hat{i} + by\hat{j} + cz\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz)$$

$= a+b+c$

$$\therefore \int\limits_S \vec{F}.\hat{n}\,ds = \int\limits_V (a+b+c)dV$$

$= (a+b+c)V$

Now V= volume of sphere of unit radius

$=\frac{4}{3}\pi.1^3$

$=\frac{4}{3}\pi$

$\therefore \int_S (ax\hat{i}+by\hat{j}+cz\hat{k}).ds$

$=(a+b+c).\ \frac{4}{3}\pi$

$=\frac{4}{3}(a+b+c)\ \pi$. Henceproved.

**Example 4:-** find $\iint_S \vec{F}.\hat{n}\,ds$ where $\vec{F}=(2x+3z)\,\hat{i}-(xz+y)\,\hat{j}+(y^2+2z)\ \hat{k}$ and S is the surface of the sphere having centre at (3, -1, 2) and radius 3.

(U.P.T.U 2001, 2005).

**Solution:-** We have by Gauss divergence theorem

$\int_S \vec{F}.\hat{n}\,ds=\int_V \text{div}\,\vec{F}\,dv$

Now div $\vec{F}=\nabla.\vec{F}$

$=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).\left\{(2x+3z)\hat{i}-(xz+y)\hat{j}+(y^2+2z)\hat{k}\right\}$

$=2-1+2$

$=3$

$\therefore \int_S \vec{F}.\hat{n}\,ds=\iiint \text{div}\vec{F}\,dv$

$=\iiint_V 3dv$

$=3\iiint_V dv$

$=3V$

But V is the volume of a sphere of radius 3

$\therefore V=\frac{4}{3}\pi\,(3)^3=36\pi$

Hence

$\int_S \vec{F}.\hat{n}\,ds=3\times 36\pi$

$=108\,\pi$ Answer

**Example 5:-** The vector field $\vec{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$ is defined over the volume of the cuboid given by $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$, enclosing the surface S. Evaluate the surface integral $\iint \vec{F}.ds$

(U.P.T.U. 2002)

**Solution :-** By Gauss divergence theorem

$$\int_S \vec{F}.\hat{n}\,ds = \int_V \operatorname{div}\vec{F}\,dv$$

$$= \iiint_V \nabla.\vec{F}\,dv$$

where V is the volume of the cuboid enclosing the surface S

$$= \iiint_V \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(x^2\hat{i} + z\hat{j} + yz\hat{k}\right)dv$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(yz)\right]dx\,dy\,dz$$

$$= \int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c} (2x + y)dx\,dy\,dz$$

$$= \int_0^a dx\int_0^b dy\int_0^c (2x + y)dz$$

$$= \int_0^a dx\int_0^b [2xz + yz]_0^c\,dy$$

$$= \int_0^a dx\int_0^b (2xc + yc)\,dy$$

$$= c\int_0^a dx\int_0^b (2x + y)\,dy$$

$$= c\int_0^a \left(2xy + \frac{y^2}{2}\right)_0^b dx$$

$$= c\int_0^a \left(2bx + \frac{b^2}{2}\right)dx$$

$$= c\left[\frac{2bx^2}{2} + \frac{b^2x}{2}\right]_0^a$$

$$= c\left(a^2b + \frac{ab^2}{2}\right)$$

$$= abc\left(a+\frac{b}{2}\right) \text{Answer}$$

**Example 6:-** Use Gauss divergence theorem to evaluate the surface integral $\iint_S (xdydz + ydzdx + zdxdy)$ where S is the portion of the plane x+2y+3z = 6 which lies in the first octant.

(U.P.T.U 2004)

**Solution :-** By Gauss's divergence theorem $\int_S \vec{F}.\hat{n}\,ds = \int_V \operatorname{div}\vec{F}\,dv$

$$\text{or } \iint_S \vec{F}.\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dv \qquad \text{(i)}$$

Here $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$, $\hat{n} = dydz\,\hat{i} + dzdx\,\hat{j} + dxdy\,\hat{k}$

$$\text{Now } \operatorname{div}\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$=1+1+1$

$= 3$

From (i) we have

$$\iint_S (xdydz + ydzdx + zdxdy) = \iiint_V 3\,dv$$

$$= 3\int_{x=0}^{6}\int_{y=0}^{\frac{6-x}{2}}\int_{z=0}^{\frac{6-x-2y}{3}} dz\,dy\,dx$$

$$= 3\int_{x=0}^{6}\int_{y=0}^{\frac{6-x}{2}} (z)_0^{\frac{6-x-2y}{3}} dy\,dx$$

$$= 3\int_{x=0}^{6}\int_{y=0}^{\frac{6-x}{2}} \left(\frac{6-x-2y}{3}\right) dy\,dx$$

$$= \int_{x=0}^{6} \left\{(6-x)y - y^2\right\}_0^{\frac{6-x}{2}} dx$$

$$= \int_{x=0}^{6} \left\{\frac{(6-x)^2}{2} - \frac{(6-x)^2}{4}\right\} dx$$

$$= \int_{x=0}^{6} \frac{(6-x)^2}{4} dx$$

$$= \frac{1}{4}\left[\frac{(6-x)^3}{(-3)}\right]_0^6$$

$$= \frac{1}{12}(216) = 18 \text{ Answer}$$

**Example 7:-** Use divergence theorem to evaluate $\iint\limits_S \vec{F}.\hat{n}\,ds$, where $\vec{F} = x^3\,\hat{i} + y^3\,\hat{j} + z^3\,\hat{k}$ and S is the surface of the sphere $x^2+y^2+z^2 = a^2$

(P.T.U. 2004)

**Solution:-** Let V be the volume bounded by the surface S

By Gauss divergence theorem, we have

$$\therefore \int\limits_S \vec{F}.\hat{n}\,ds = \iiint \text{div}\vec{F}\,dv \qquad \text{(i)}$$

Now div $\vec{F} = \nabla.\vec{F}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(x^3\hat{i} + y^3\hat{j} + z^3\hat{k}\right)$$

$$= 3(x^2+y^2+z^2) \qquad \text{(ii)}$$

$$\therefore \iint\limits_S \vec{F}.\hat{n}\,ds = 3\iiint\limits_V (x^2 + y^2 + z^2)\,dv$$

Changing to spherical polar Co-ordinates by putting x= r sinθ cos ϕ,

y = r sin θ sin ϕ, z = cos θ

dv = $r^2$ sinθ dr dθ dϕ

The limits of integration will be 0 to a, 0 to π and 0 to 2π

$$\therefore 3\iiint (x^2 + y^2 + z^2)\,dv = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{a} r^2.r^2 \sin\theta\,dr\,d\theta\,d\phi$$

$$= 3\int_0^{2\pi}\int_0^{\pi/2}\left[\frac{r^5}{5}\right]_0^a \sin\theta\,d\phi$$

$$= \frac{3}{5}a^5\int_0^{2\pi} d\phi\,[-\cos\theta]_0^{\pi}$$

$$= \frac{3}{5}a^5\,[\phi]_0^{2\pi} \times 2$$

$$= \frac{12}{5}\pi a^5 \text{ Answer}$$

**Example 8:-** Verify Divergence theorem for $\vec{F} = (x^2\text{-}yz)\,\hat{i} + (y^2\text{-}zx)\,\hat{j} + (z^2\text{-}xy)\,\hat{k}$, taken over the rectangular parallelopiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$,

(U.P.T.U. 2002, V.T.U. 200 M.D.U. 2006, P.T.U. 2003)

**Solution :**

As div $\vec{F} = \nabla.\vec{F}$

$$= \frac{\partial}{\partial x}\left(x^2 - yz\right) + \frac{\partial}{\partial y}\left(y^2 - zx\right) + \frac{\partial}{\partial z}\left(z^2 - xy\right)$$

$$= 2(x+y+z)$$

Therefore, volume integral

$$= \int_V \operatorname{div} \bar{F}\, dv = 2\iiint_V (x+y+z)\, dv$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c} (x+y+z)\, dx\, dy\, dz$$

$$= 2\int_0^a dx \int_0^b dy \int_0^c (x+y+z)\, dz$$

$$= 2\int_0^a dx \int_0^b dy \left( xz + yz + \frac{z^2}{2} \right)_0^C$$

$$= 2\int_0^a dx \int_0^b \left( cx + cy + \frac{c^2}{2} \right)$$

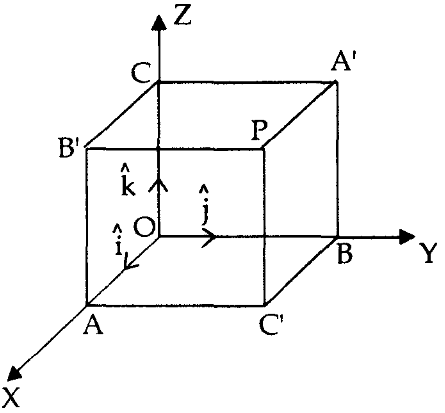
$$= 2\int_0^a dx \left( cxy + \frac{cy^2}{2} + \frac{c^2 y}{2} \right)_0^b$$

$$= 2\int_0^a dx \left( bcx + \frac{b^2 c}{2} + \frac{bc^2}{2} \right)$$

$$= 2\left[ \frac{bcx^2}{2} + \frac{b^2 cx}{2} + \frac{bc^2 x}{2} \right]_0^a$$

$= (a^2bc+ab^2c+abc^2)$

$= abc\,(a+b+c)$ (i)



To evaluate the surface integral, divided the closed surface S of the rectangular parallelopiped into six parts i.e.

$S_1$ : The face OAC'B

$S_2$ : The face CB'PA'

$S_3$ : The face OBA'C
$S_4$ : The face AC'PB'
$S_5$ : The face OCB'A
$S_6$ : The face BA'PC'

Also $\iint_S \vec{F}.\hat{n}\,ds = \iint_{S_1} \vec{F}.\hat{n}ds + \iint_{S_2} \vec{F}.\hat{n}ds + \iint_{S_3} \vec{F}.\hat{n}ds + \iint_{S_4} \vec{F}.\hat{n}ds + \iint_{S_5} \vec{F}.\hat{n}ds + \iint_{S_6} \vec{F}.\hat{n}ds$

on $S_1$ $(z = 0)$, we have $\hat{n} = -\hat{k}$ , $\vec{F} = x^2\hat{i} + y^2\hat{j} - xy\hat{k}$ so that

$\vec{F}.\hat{n} = \left(x^2\hat{i} + y^2\hat{j} - xy\hat{k}\right).\left(-\hat{k}\right)$

$= xy$

$\therefore \iint_S \vec{F}.\hat{n}\,ds = \int_0^b\int_0^a xy\,dx\,dy$

$= \int_0^b \left[y\frac{x^2}{2}\right]_0^a dy = \frac{a^2}{2}\int_0^b y dy = \frac{a^2b^2}{4}$

on $S_2$ $(z = c)$, we have $\hat{n} = \hat{k}$

$\vec{F} = (x^2 - cy)\,\hat{i} + (y^2 - cx)\,\hat{j} + (c^2 - xy)\,\hat{k}$

so that $\vec{F}.\hat{n}ds = \left[\left(x^2 - cy\right)\hat{i} + \left(y^2 - cx\right)\hat{j} + \left(c^2 - xy\right)\hat{k}\right].\hat{k}$

$= c^2 - xy$

$\therefore \iint_{S_2} \vec{F}.\hat{n}\,ds = \int_0^b\int_0^a \left(c^2 - xy\right)dx\,dy = \int_0^b \left(c^2a - \frac{a^2y}{2}\right)dy$

$\therefore \iint_{S_4} \vec{F}.\hat{n}\,ds = \int_0^c\int_0^b \left(a^2 - yz\right)dy\,dz = \int_0^c \left(a^2b - \frac{b^2z}{2}\right)dz$

$= abc^2 - \frac{a^2b^2}{4}$

on $S_3$ $(x = 0)$ we have $\hat{n} = -\hat{i}$ , $\vec{F} = -yz\,\hat{i} + y^2\,\hat{j} + z^2\,\hat{k}$ ,

so that $\vec{F}.\hat{n} = \left(-yz\hat{i} + y\hat{j} + z^2\hat{k}\right).\left(-\hat{i}\right) = yz$

$\therefore \iint_{S_3} \vec{F}.\hat{n}\,ds = \int_0^c\int_0^b yz\,dy\,dz = \int_0^c \frac{b^2}{2}x dz = \frac{b^2c^2}{4}$

on $S_4$ $(x = a)$, we have $\hat{n} = i$ , $\vec{F} = (a^2 - yz)\,\hat{i} + (y^2 - az)\,\hat{j} + (z^2 - ay)\,\hat{k}$ ,

so that $\vec{F}.\hat{n} = \left[\left(a^2 - yz\right)\hat{i} + \left(y^2 - az\right)\hat{j} + \left(z^2 - ay\right)\hat{k}\right].\hat{i}$

$= a^2 - yz$

$\therefore \iint_{S_4} \vec{F}.\hat{n}\,ds = \int_0^c\int_0^b (a^2 - yz)dy\,dz = \int_0^c \left(a^2b - \frac{b^2z}{2}\right)dz$

$$= a^2 bc - \frac{b^2c^2}{4}$$

on $S_5$ $(y=0)$ we have $\hat{n} = -\hat{j}$, $\vec{F} = x^2\hat{i} - zx\hat{j} + z^2\hat{k}$,

so that = $\vec{F}\,.\,\hat{n} = (x^2\hat{i} - zx\hat{j} + z^2\hat{k})\,(-\hat{j}) = zx$

$$\therefore \iint_{S_5} \vec{F}.\hat{n}\,ds = \int_0^a\int_0^c zx\,dz\,dx = \int_0^c \frac{c^2}{2}x\,dx = \frac{a^2c^2}{4}$$

on $S_6$ $(y = b)$, we have $\hat{n} = \hat{j}$, $\vec{F} = (x^2 - bz)\hat{i} + (b^2 - zx)\hat{j} + (z^2 - bx)\hat{k}$,

so that $\vec{F}.\hat{n} = \left[(x^2 - bz)\hat{i} + (b^2 - zx)\hat{j} + (z^2 - bx)\hat{k}\right].\hat{j}$

$= b^2 - zx$

$$\therefore \iint_{S_6} \vec{F}.\hat{n}\,ds = \int_0^a\int_0^c (b^2 - zx)\,dz\,dx$$

$$= \int_0^a \left(b^2c - \frac{c^2}{2}x\right)dx$$

$$= ab^2c - \frac{a^2c^2}{4}$$

$$\therefore \iint_S \vec{F}.\hat{n}\,ds = \frac{a^2b^2}{4} + abc^2 - \frac{a^2b^2}{4} + \frac{b^2c^2}{4} + a^2bc - \frac{b^2c^2}{4} + \frac{a^2c^2}{4} + ab^2c - \frac{a^2c^2}{4}$$

$= abc\,(a+b+c)$ (ii)

The equality of (i) and verifies divergence theorem.

**Example 9:-** Verify divergence theorem for $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ take over the region bounded by the cylinder $x^2+y^2 = 4$, $z = 0$, $z = 3$

**Solution:-** Since div $\vec{F} = \nabla.\vec{F}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)$$

$= 4 - 4y + 2z$

Therefore

$$\iiint_V \text{div}\,\vec{F}.dv = \iiint_V (4 - 4y + 2z)\,dx\,dy\,dz$$

$$= \int_{x=-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3}(4-4y+2z)dx\,dy\,dz$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[4z-4yz+z^2\right]_0^3 dx\,dy$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(12-12y+9)dx\,dy$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(21-12y)dx\,dy$$

$$= 21\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}dx\,dy - 12\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}y\,dx\,dy$$

$$= 21\times 4\int_{x=0}^{2}\int_{y=0}^{\sqrt{4-x^2}}dx\,dy - 12[0]$$

Since y is an odd function in the second integral

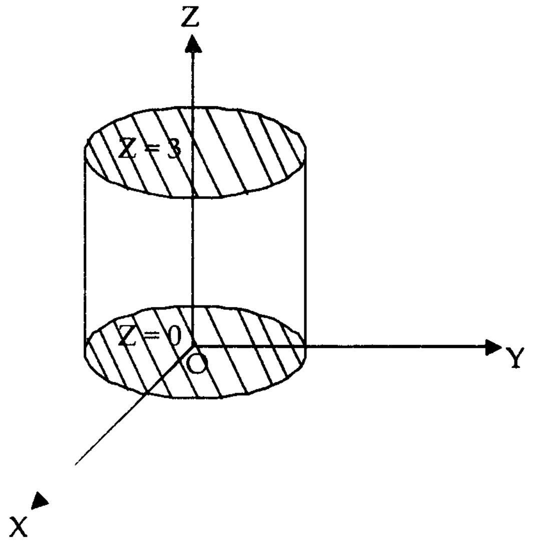
$$= 84\int_{x=0}^{2}(y)_0^{\sqrt{4-x^2}}dx$$

$$= 84\int_0^2\sqrt{\left(2^2-x^2\right)}\,dx$$

$$= 84\left[\frac{x}{2}\sqrt{\left(2^2-x^2\right)}+\frac{1}{2}.2^2\sin^{-1}\frac{x}{2}\right]_0^2$$

$= 84\ [2\sin^{-1}1]$

$= 84\ [2\ (\pi/2)] = 84\pi$



To evaluate the surface integral, divide the closed surface S of the cylinder into three parts,

$S_1$ : The circular base in the plane z = 0

$S_2$ : The circular top in the plane z = 3

$S_3$ : The curved surface of the cylinder given by the equation $x^2+y^2 = 4$

Also $\iint_S \vec{F}.\hat{n}\,ds = \iint_{S_1} \vec{F}.\hat{n}\,ds + \iint_{S_2} \vec{F}.\hat{n}\,ds + \iint_{S_3} \vec{F}.\hat{n}\,ds$

on $S_1$ (z=0). we have $\hat{n} = -\hat{k}$ $\vec{F} = 4x\hat{i} - 2y^2\hat{j}$

so that $\vec{F}.\hat{n} = \left(4x\hat{i} - 2y^2\hat{j}\right).\left(-\hat{k}\right) = 0$

$\therefore \iint_{S_1} \vec{F}.\hat{n}\,ds = 0$

on $S_2$ (z=3), we have $\hat{n} = \hat{k}$, $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + 9\hat{k}$

so that $\vec{F}.\hat{n} = \left(4x\hat{i} - 2y^2\hat{j} + 9\hat{k}\right).\hat{k}$

$= 9$

$\therefore \iint_{S_2} \vec{F}.\hat{n}\,ds = \iint_{S_2} 9\,dx\,dy$

$= 9 \times$ area of surface $S_2$

$= 9(\pi.\ 2^2) = 36\,\pi$

Where $S_2$ is the area of circle of radius 2

on $S_3$, $x^2+y^2=4$

A vector normal to the surface $S_3$ is given by $\nabla(x^2+y^2) = 2x\hat{i} + 2y\hat{j}$

$\therefore \hat{n}$ = a unit vector normal to surface $S_3$

$$= \frac{2x\hat{i}+2y\hat{j}}{\sqrt{(2x)^2+(2y)^2}} = \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}} \qquad \therefore \text{unit normal} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4\left(x^2+y^2\right)}}$$

$$= \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4\times4}}$$

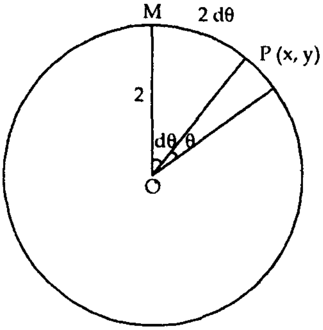
$$= \frac{x\hat{i}+y\hat{j}}{2}$$

$\because x^2+y^2 = 4$

$$\therefore \vec{F}.\hat{n} = \left(4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}\right).\left(\frac{x\hat{i}+y\hat{j}}{2}\right)$$

$= 2x^2 - y^2$

Also, on $S_3$, i.e. $x^2+y^2=4$, $x = 2\cos\theta$, $y = 2\sin\theta$ and $ds = 2\,d\theta\,dz$



As surface area of element P(width 2θ and height dz) is 2dθ dz.
To cover the whole area S₃, z varies from 0 to 3 and θ varies from 0 to 2π

$$\therefore \iint_{S_3} \vec{F}.\hat{n}\,ds = \int_0^{2\pi}\int_0^3 \left[2(2\cos\theta)^2 - (2\sin\theta)^3\right] 2d\theta dz$$

$$= \int_0^{2\pi} 16\left(\cos^2\theta - \sin^3\theta\right)3d\theta$$

$$= 48\int_0^{2\pi}\left(\cos^2\theta - \sin^3\theta\right)d\theta$$

$$= 48\,\pi$$

$$\therefore \iint_S \vec{F}.\hat{n}\,ds = 0 + 36\pi + 48\pi$$

$= 84\pi$ (ii)

Therefore from (i) and (ii) Thus $\iint_S \vec{F}.\hat{n}\,ds = \iiint_V \text{div}\,\vec{F}\,dv$ Thus divergence theorem is verified.

**Example 10 :** Verify Gauss divergence theorem for the surface $\vec{F} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$ taken over the cube bounded by the planes x =0, x =1, y = 0 y =1, z =0, z =1.

**Solution :** Given $\vec{F} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$ (i)

$$\therefore \text{div}\ \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial x} + \hat{k}\frac{\partial}{\partial x}\right).\left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial x}(yz)$$

$= 4z - 2y + y$

$= 4z - y$

By Gauss theorem, we have

$$\int_S F.n\,ds = \int_V \text{div}\,\vec{F}\,dv$$

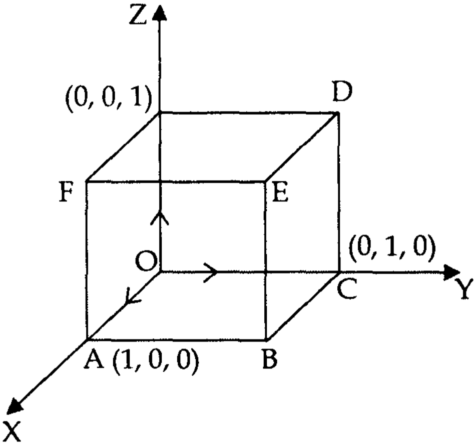
$$= \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}(4z - y)\,dx\,dy\,dz$$

$$= \int_{0}^{1}\int_{0}^{1}\left(2z^2 - yz\right)_{z=0}^{1}dx\,dy$$

$$= \int_{0}^{1}\int_{0}^{1}(2 - y)\,dx\,dy$$

$$= \int_{x=0}^{1}\left(2y - \frac{1}{2}y^2\right)_{y=0}^{1}dx$$

$$= \int_{0}^{1}\left(2 - \frac{1}{2}\right)dx = \frac{3}{2}[x]_0^1 = \frac{3}{2} \qquad \text{(ii)}$$



To evaluate $\iint_S \vec{F}.\hat{n}\,ds$

Here S is the surface of the cube bounded by the six plane surfaces
Over the face OABC

z =0, dz =0, $\hat{n} = -\hat{k}$, ds = dxdy

$$\therefore \iint \vec{F}.\hat{n}\,ds = \int_0^1\int_0^1\left(-y^2 j\right).(-k)\,dx\,dy = 0 \qquad \text{(iii)}$$

Over the face BCDE,

y = 1, dy =0, $\hat{n} = \hat{j}$, ds = dx dz

$$\therefore \iint \vec{F}.\hat{n}\,ds = \int_0^1\int_0^1\left(4xz\hat{i} - \hat{j} + z\hat{k}\right).\hat{j}\,dx\,dz$$

$$= -\int_0^1\int_0^1 dx\,dz$$

$$= -(x)_0^1 (z)_0^1 = -1 \qquad \text{(iv)}$$

Over the face DEFG,

$z = 1$, $dz = 0$, $\hat{n} = \hat{k}$, $ds = dx\,dy$

$$\therefore \iint \vec{F}.\hat{n}\,ds = \int_0^1\int_0^1 \left(4x\hat{i} - y^2\hat{j} + y\hat{k}\right).\hat{k}\,dx\,dy$$

$$= \int_0^1\int_0^1 y\,dx\,dy$$

$$= \int_0^1 dx \int_0^1 y\,dy$$

$$= (x)_0^1 \left(\frac{y^2}{2}\right)_0^1 = \frac{1}{2} \qquad \text{(v)}$$

Over the face AOGE,

$y = 0$, $dy = 0$, $\hat{n} = -\hat{j}$, $ds = dx\,dz$

$$\therefore \iint \vec{F}.\hat{n}\,ds = \int_0^1\int_0^1 \left(4xz\hat{i}\right).\left(-\hat{j}\right)dx\,dz$$

$$= 0 \qquad \text{(vi)}$$

Over the face OCDG,

$x = 0$, $dx = 0$, $\hat{n} = -\hat{i}$, $ds = dy\,dz$

$$\therefore \iint \vec{F}.\hat{n}\,ds = \int_0^1\int_0^1 \left(-y^2\hat{j} + yz\hat{k}\right).\left(-\hat{i}\right)dy\,dz$$

$$= 0 \qquad \text{(vii)}$$

Over the face ABEF,

$x = 1$, $dx = 0$, $\hat{n} = \hat{i}$, $ds = dy\,dz$

$$\therefore \iint \vec{F}.\hat{n}\,ds = \int_0^1\int_0^1 \left(4z\hat{i} - y^2\hat{j} + yz\hat{k}\right).\hat{i}\,dy\,dz$$

$$= \int_0^1\int_0^1 4z\,dy\,dz$$

$$= \int_0^1 dy \int_0^1 4z\,dz$$

$$= (y)_0^1 \left(2z^2\right)_0^1 = 2 \qquad \text{(viii)}$$

Adding equations (iii), (iv), (v), (vi), (vii) & (viii) we get
Over the whole surface S,

$$\iint_S \vec{F}.\hat{n}\,ds = 0 -1+\frac{1}{2}+0+0+2$$

$$=\frac{3}{2} \qquad \text{(ix)}$$

From equations (ii) and (ix) we have

$$\iint_S \vec{F}.\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dv$$

Hence the divergence theorem is verified.

**Example 11:** By transforming to a triple integral, evaluate

$I=\iint_S \left(x^3\,dydz + x^2y\,dzdx + x^2z\,dxdy\right)$, where S is the closed surface bounded by the planes z = 0, z = b and the cylinder $x^2+y^2=a^2$

(U.P.T.U. 2006)

**Solution:-** By Gauss divergence theorem, the required surface integral I is equal to the volume integral

$$\iiint_V \left[\frac{\partial}{\partial x}(x^3)+\frac{\partial}{\partial y}(x^2y)+\frac{\partial}{\partial z}(x^2z)\right]dv$$

$$=\int_{z=0}^{b}\int_{y=-a}^{a}\int_{x=-\sqrt{(a^2-y^2)}}^{\sqrt{(a^2-y^2)}}\left(3x^2+x^2+x^2\right)dz\,dy\,dx$$

$$=4\times 5\int_{z=0}^{b}\int_{y=0}^{a}\int_{x=0}^{\sqrt{(a^2-y^2)}}x^2\,dz\,dy\,dx$$

$$=20\int_{z=0}^{b}\int_{y=0}^{a}\left[\frac{x^3}{3}\right]_{x=0}^{\sqrt{(a^2-y^2)}}dz\,dy$$

$$=\frac{20}{3}\int_{y=0}^{a}\int_{z=0}^{b}\left(a^2-y^2\right)^{3/2}dz\,dy$$

$$=\frac{20}{3}\int_{y=0}^{a}\left[\left(a^2-y^2\right)^{3/2}z\right]_{z=0}^{b}dy$$

$$=\frac{20}{3}\int_{y=0}^{a}b\left(a^2-y^2\right)^{3/2}dy$$

Putting y = a sin t, dy = a cos t dt

$$\therefore I=\frac{20}{3}b\int_0^{\pi/2}a^2\cos^3 t\left(a\cos t\right)dt$$

$$=\frac{20}{3}a^4b\int_0^{\pi/2}\cos^4 t\,dt$$

$$=\frac{20}{3}a^4b\frac{3}{4.2}\,\frac{\pi}{2}=\frac{5}{4}\pi a^4b \text{ Answer}$$

**Stoke's Theorem :** If $\vec{F}$ is any continuously differentiable vector point function and S is a surface bounded by a curve C, then

$$\int_C \vec{F}.d\vec{r} = \int_S \text{Curl}\,\vec{F}.\hat{n}\,ds$$

Where $\hat{n}$ is unit outward drawn normal vector at any point of the surface S.

**Example 1:** Verify stoke's theorem for $\vec{F} = (2x-y)\,\hat{i} - yz^2\,\hat{j} - y^2z\,\hat{k}$ where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ and C is its boundary.

(U.P.T.U. 2001, I.A.S 2004, B.P.S.C 2001)

**Solution :** The boundary C of S is a circle in the xy plane of radius unity and centre at the origin. The parametric equations of C will be x= cos t, y = sin t, z= 0, $0 \le t \le 2\pi$ consequently, the position vector of any point on the circle is $\vec{r} = x\hat{i} + y\hat{j}$. Hence.

$$\int_c \vec{F}.d\vec{r} = \int_c \left\{(2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}\right\}.\left(\hat{i}dx + \hat{j}dy\right)$$

$$= \int_c (2x-y)dx - \int_c yz^2\,dy$$

$$= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)dt$$

$$= \int_0^{2\pi} -(2\cos t - \sin t)\sin t\,dt$$

$$= \int_0^{2\pi} \sin^2 t\,dt - \int_0^{2\pi} \sin 2t\,dt$$

$$= 4\int_0^{\pi/2} \sin^2 t\,dt + \left(\frac{\cos 2t}{2}\right)_0^{2\pi}$$

$$= \frac{4\,\Gamma\frac{1}{2}\,\Gamma\frac{3}{2}}{2\,\Gamma 2} + \frac{1}{2}(1-1)$$

$$= \frac{4.\frac{\pi}{2}}{2}$$

$$= \pi \qquad \text{(i)}$$

Also

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= \hat{i}\,(-2\,yz + 2yz) + \hat{j}\,(0) + \hat{k}\,(1)$$

$= \hat{k}$

Then $\int_S \text{curl}\,\vec{F}.\hat{n}\,ds = \int_S \hat{k}.\hat{n}\,ds$

$= \int_s ds$

$$\because ds = \frac{dx\,dy}{\hat{n}.\hat{k}}$$

$= \iint_R dx\,dy$

where R is the projection of S on the xy plane.
Therefore

$$\iint_R dx\,dy = \int_{x=-1}^{x=1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx\,dy$$

$$= 4\int_0^1\int_0^{\sqrt{1-x^2}} dx\,dy$$

$$= 4\int_0^1 \sqrt{1-x^2}\,dx$$

$$\because \int\sqrt{(a^2-x^2)}\,dx = \frac{x}{2}\sqrt{(a^2-x^2)} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$= \pi \qquad \text{(ii)}$$

From (i) and (ii) we get

$\int_s \text{curl}\,\vec{F}.\hat{n}\,ds = \int_c \vec{F}.d\vec{r}$

Hence stoke's theorem is verified.

**Example 2 :** Verify stoke's theorem when $\vec{F} = y\,\hat{i} + z\,\hat{j} + x\,\hat{k}$ and surface S is the part of the sphere $x^2+y^2+z^2=1$ above the xy plane.

(U.P.P.C.S. 2003)

**Solution :** Stoke's theorem is

$\int_C \vec{F}.d\vec{r} = \int_S \text{curl}\,\vec{F}.\hat{n}\,ds$

where C is the unit circle $x^2+y^2=1$, $z=0$ and $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$

$\Rightarrow d\,\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$\therefore \vec{F}.d\vec{r} = (y\hat{i} + z\hat{j} + x\hat{k}).(dx\hat{i} + dy\hat{j} + dz\hat{k})$

$= ydx + zdy + xdz$

So

$\int_c \vec{F}.d\vec{r} = \int_c ydx + \int_c zdy + \int_c xdz$

The parametric equations of the circle is

$x = \cos t$, $y= \sin t$ , $0 \le t \le 2\pi$ $z = 0$, $dz = 0$

Therefore

$$\int_c \vec{F}.d\vec{r} = \int_c y\,dx$$

$$= \int_0^{2\pi} \sin t(-\sin t)dt$$

$\because$ $x = \cos t$, $y = \sin t$

$\Rightarrow dx = -\sin t\, dt$

$$= -\int_0^{2\pi} \sin^2 t dt$$

$$= -4\int_0^{2\pi} \sin^2 t dt$$

$$= -4\frac{\lceil\left(\frac{3}{2}\right) \lceil\left(\frac{1}{2}\right)}{2\lceil(2)}$$

$$= -4\frac{\frac{1}{2}\sqrt{\pi}\sqrt{\pi}}{2}$$

$= -\pi$ (i)

Again

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$= -\hat{i} - \hat{j} - \hat{k}$$

Let $\hat{n}$ be the outward unit normal to the surface $x^2+y^2+z^2 =1$ at point $(x,y,z)$, then

$\phi = x^2 +y^2 + z^2 -1$

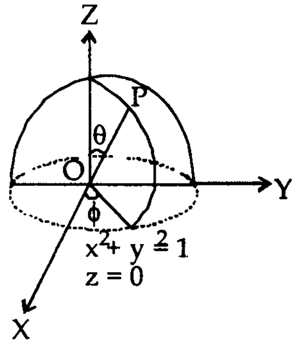
$$\frac{\partial \phi}{\partial x} =2x, \frac{\partial \phi}{\partial y} =2y, \frac{\partial \phi}{\partial z} =2z$$

$$\therefore \hat{n} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{(2x)^2 + (2y)^2 + (2z)^2}}$$

$$= x\hat{i} + y\hat{j} + z\hat{k}$$

$\therefore$ $x^2+y^2+z^2 =1$

$\therefore$ $\text{Curl}\,\vec{F}.\hat{n} = - (x+y+z)$



Using spherical polar coordinates

$x = r\sin\theta\cos\phi = \sin\theta\cos\phi \qquad \because r = 1$

$y = r\sin\theta\sin\phi = \sin\theta\sin\phi$

$z = r\cos\theta = \cos\theta$

and $ds = \sin\theta\, d\theta\, d\phi$

In first octant $\theta = 0$ to $\pi/2$, $\phi = 0$ to $2\pi$

$\text{curl}\,\vec{F}.\hat{n} = -(\sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta)$

$$\therefore \int_S \text{curl}\vec{F}.\hat{n}ds = -\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}(\sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta)\sin\theta\, d\theta\, d\phi$$

$$= -\int_0^{\pi/2}\left[\sin\theta\,\sin\phi - \sin\theta\cos\phi + \phi\cos\theta\right]_0^{2\pi}\sin\theta\, d\theta$$

$$= -2\pi\int_0^{\pi/2}\cos\theta\sin\theta\, d\theta$$

$$= -\pi\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2}$$

$= -\pi$ (ii)

From (i) & (ii) we get

$$\int_s \text{curl}\,\vec{F}.\hat{n}\, ds = \int_c \vec{F}.d\vec{r}$$

Hence stoke's theorem is verified.

**Example 3:** Verify stoke's theorem for the function $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$ where C is the unit circle in xy- plane bounding the hemisphere $Z = \sqrt{(1 - x^2 - y^2)}$

(I.A.S. 1993, U.P.P.C.S. 1999, U.P.T.U.(C.O) 2003)

**Solution :** Stoke's theorem is $\int_C \vec{F}.d\vec{r} = \int_S \text{Curl}\,\vec{F}.\hat{n}\, ds$ where C is the unit circle

$x^2+y^2=1$, $z = 0$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\Rightarrow d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$\vec{F}.d\vec{r} = \left(z\hat{i} + x\hat{j} + y\hat{k}\right).\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$
$$= zdx+xdy+ydz$$
so $\int_c \vec{F}.d\vec{r} = \int_c zdx + \int_c xdy + \int_c ydz$

The parametric equations of the curve is x= cos t, y = sin t, $0 \le t \le 2\pi$, z = 0, dz = 0

Therefore,

$$\int_c \vec{F}.d\vec{r} = \int_c x\,dy$$
$$= \int_0^{2\pi} \cos t \cos t\, dt$$

$\because$ x = cos t

y = sin t

$\Rightarrow$ dy = cos t dt

$$= \int_0^{2\pi} \cos^2 t\,dt$$
$$= 4\int_0^{\pi/2} \cos^2 t\,dt$$
$$= \frac{4.\Gamma\left(\frac{3}{2}\right)\Gamma\frac{1}{2}}{2\Gamma 2}$$
$$= \frac{4.\frac{1}{2}.\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{2}$$
$$= \sqrt{\pi}\sqrt{\pi}$$
$$= \pi$$

i.e. $\int_c \vec{F}.d\vec{r} = \pi$ (i)

Again curl $\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$

$$= \hat{i} + \hat{j} + \hat{k}$$

Let $\hat{n}$ be the outward unit normal to the surface $x^2+y^2+z^2 = 1$ at point (x,y, z), Then

$\phi \equiv x^2+y^2+z^2 -1$

$$\therefore \hat{n} = \frac{\text{grad}\phi}{|\text{grad}\phi|}$$

$$= \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{(2x)^2 + (2y)^2 + (2z)^2}}$$

$= x\hat{i} + y\hat{j} + z\hat{k}$ $\quad\because x^2+y^2+z^2 = 1$

Therefore, curl $\vec{F}.\hat{n} = (\hat{i} + \hat{j} + \hat{k}).(x\hat{i} + y\hat{j} + z\hat{k})$

$= x+y+z$

using spherical polar coordinates

$x = r \sin\theta \cos\phi = \sin\theta \cos\phi$ $\quad\because r = 1$

$y = r \sin\theta \sin\phi = \sin\theta \sin\phi$

$z = r \cos\theta = \cos\theta$

and $ds = \sin\theta \, d\theta \, d\phi$

In first octant $\theta = 0$ to $\pi/2$ and $\phi = 0$ to $2\pi$

curl $\vec{F}.\hat{n} = \sin\theta \cos\phi + \sin\theta \sin\phi + \cos\theta$

$$\therefore \int_S \text{curl}\,\vec{F}.\hat{n}\,ds = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} (\sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta)\sin\theta d\theta d\phi$$

$$= \int_0^{\pi/2} [\sin\theta\sin\phi - \sin\theta\cos\phi + \phi\cos\theta]_0^{2\pi} \sin\theta d\theta$$

$$= 2\pi \int_0^{\pi/2} \cos\theta\sin\theta d\theta$$

$$= \pi \int_0^{\pi/2} \sin 2\theta d\theta$$

$$= \pi\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$= -\frac{\pi}{2}[\cos\pi - \cos 0]$$

$$= -\frac{\pi}{2}[-1 - 1]$$

$$= -\frac{\pi}{2}(-2)$$

$= \pi$ (ii)

From (i) and (ii) we get

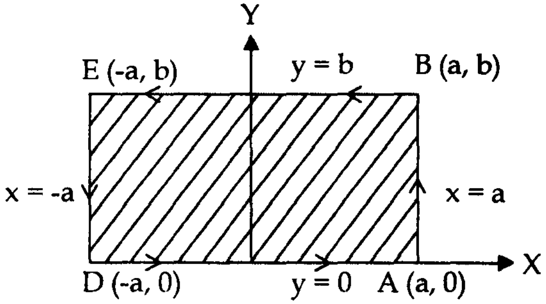
$$\int_s \text{curl}\,\vec{F}.\hat{n}ds = \int_c \vec{F}.d\vec{r}$$

Which verifies Stoke's theorem

**Example 4:** Verify stoke's theorem for $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ taken round the rectangle bounded by the lines $x = \pm a$, $y = 0$, $y = b$

(U.P.T.U. 2003, U.P.P.C.S. 1997)

**Solution :**



Let C denote the boundary of the rectangle ABED, then

$$\oint_c \vec{F}.d\vec{r} = \oint_c \left[(x^2+y^2)\hat{i} - 2xy\hat{j}\right].\left(\hat{i}dx + \hat{j}dy\right)$$

$$= \oint_c \left[(x^2+y^2)dx - 2xy\,dy\right]$$

The curve C consists of four lines AB, BE, ED and DA.

Along AB, $x=a$, $dx = o$ and y varies from 0 to b

$$\therefore \int_{AB} \left[(x^2+y^2)dx - 2xy\,dy\right] = \int_0^b -2ay\,dy$$

$$= -a\left[y^2\right]_0^b = -ab^2 \qquad \text{(i)}$$

Along BE, $y = b$, $dy = 0$ and x varies from a to -a

$$\therefore \int_{BE} \left[(x^2+y^2)dx - 2xy\,dy\right] = \int_a^{-a} (x^2+b^2)dx$$

$$= \left[\frac{x^3}{3} + b^2x\right]_a^{-a}$$

$$= -\frac{2a^2}{3} - 2\,ab^2 \qquad \text{(ii)}$$

Along ED, $x = -a$, $dx = 0$ and y varies from b to 0

$$\therefore \int_{ED} \left[(x^2+y^2)dx - 2xy\,dy\right] = \int_b^0 2ay\,dy$$

$$= a\left[y^2\right]_b^0 = -ab^2 \qquad \text{(iii)}$$

Along DA, $y = 0$. $dy = 0$ and x varies from -a to a.

$$\therefore \int_{DA} \left[(x^2+y^2)dx - 2xy\,dy\right] = \int_{-a}^{0} x^2\,dx$$

$$= \frac{2a^3}{3} \qquad \text{(iv)}$$

Adding (i) (ii) (iii) and (iv), we get

$$\oint_c \vec{F}.d\vec{r} = -ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3}$$

$$= -4\,ab^2 \qquad \text{(v)}$$

Now,

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$= (-2y - 2y)\ \hat{k}$

$= -4y\,\hat{k}$

For the surface S , $\hat{n} = \hat{k}$

Curl $\vec{F}.\hat{n} = -\,4y\,\hat{k}\,.\,\hat{k} \;\; = -\,4y$

$$\therefore \iint_s \text{Curl}\,\vec{F}.\hat{n}\,ds = \int_0^b \int_{-a}^{a} -4y\,dx\,dy$$

$$= \int_0^b -4y\,[x]_{-a}^{a}\,dy$$

$$= -8a\int_0^b y\,dy$$

$$= -8a\left[\frac{y^2}{2}\right]_0^b$$

$$= -\,4ab^2 \qquad \text{(vi)}$$

The equality of (v) and (vi) verifies Stoke's theorem.

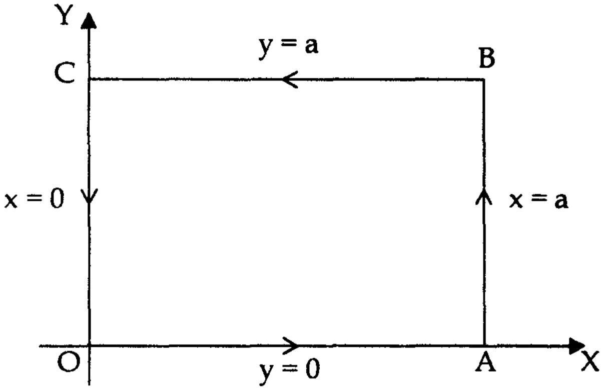
**Example 5 :** Verify Stoke's theorem for the function $\vec{F} = x^2\,\hat{i} + xy\,\hat{j}$ integrated round the square whose sides are $x = 0, y = 0, x = a, y = a$ in the plane $z = 0$.

(I.A.S. 2006)

**Solution :** Given $\vec{F} = x^2\,\hat{i} + xy\,\hat{j}$

$$\therefore \text{ Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix}$$

$$= y\ \hat{k}$$



Here $\hat{n} = k$ ($\hat{n}$ is perpendicular to xy plane)

$$\iint_s \text{Curl}\,\vec{F}.\hat{n}\,ds = \iint_s y\hat{k}.\hat{k}\,dx\,dy$$

$$= \int_0^a dx \int_0^a y\,dy$$

$$= a\left(\frac{y^2}{2}\right)_0^a = \frac{a^3}{2} \qquad \text{(i)}$$

$$\text{Now } \int_c \vec{F}.d\vec{r} = \int_c \left(x^2\hat{i} + xy\hat{j}\right).\left(dx\hat{i} + dy\hat{j}\right)$$

$$= \int_c \left(x^2\,dx + xy\,dy\right) \qquad \text{(ii)}$$

Where C is the path OABCO as shown above.

$$\int_{OABCO} \vec{F}.d\vec{r} = \int_{OA} \vec{F}.d\vec{r} + \int_{AB} \vec{F}.d\vec{r} + \int_{BC} \vec{F}.d\vec{r} + \int_{CO} \vec{F}.d\vec{r} \qquad \text{(iii)}$$

Along OA, y = 0, dy =0

$$\int_{OA} \vec{F}.d\vec{r} = \int_0^a x^2\,dx = \left(\frac{x^3}{3}\right)_0^a = \frac{a^3}{3} \qquad \text{(iv)}$$

Along AB, x= 0, dx =0

$$\int_{AB} \vec{F}.d\vec{r} = \int_0^a ay\,dy = a\left(\frac{y^2}{2}\right)_0^a = \frac{a}{2}\left(a^2\right) = \frac{a^3}{2} \qquad \text{(v)}$$

Along BC, y =a, dy =0

$$\int_{BC} \vec{F}.d\vec{r} = \int_{BC} x^2dx = \int_a^0 x^2dx = -\frac{a^3}{3} \qquad \text{(vi)}$$

Along CO, x =0, dx =0

$$\int_{CO} \vec{F}.d\vec{r} = \int_{CO} \left(x^2dx + xy\,dy\right) = 0 \qquad \text{(viii)}$$

From (iii), $\int_c \vec{F}.d\vec{r} = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0$

Using (iv) (v) (vi) & (vii)

$$= \frac{a^3}{2} \qquad \text{(viii)}$$

From Equations (i) and (viii), we have

$$\int_c F.dr = \iint_s \text{Curl}\,\vec{F}.\hat{n}\,ds$$

Hence Stoke's theorem is verified.

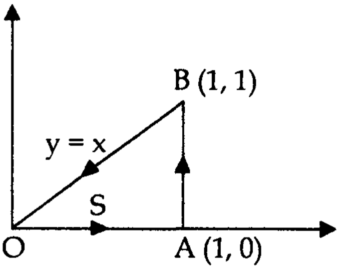
**Example 6 :** Evaluate $\oint_c \vec{F}.d\vec{r}$ by Stoke's theorem, where $\vec{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$ and C is the boundary of the triangle with vertices at (0, 0, 0), (1, 0, 0) and (1, 1, 0)

(U.P.T.U. 2001)

**Solution :** Since Z- coordinates of each vertex of the triangle is zero, therefore, the triangle lies in the xy plane and $\hat{n} = \hat{k}$

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix}$$

$= \hat{j} + 2(x-y)\,\hat{k}$



$\therefore$ Curl $\vec{F}.\hat{n} = \left[\hat{j} + 2(x-y)\hat{k}\right].\hat{k}$

$= 2(x\text{-}y)$

The equation of line OB is y = x By stoke's theorem,

$$\oint_c \vec{F}.d\vec{r} = \iint_s \text{Curl}\,\vec{F}.\hat{n}\,ds$$

$$= \int_{x=0}^{1} \int_{y=0}^{x} 2(x-y)\,dx\,dy$$

$$= 2\int_{x=0}^{1} \left[xy - \frac{y^2}{2}\right]_{y=0}^{x} dx$$

$$= 2\int_{x=0}^{1} \left(x^2 - \frac{x^2}{2}\right) dx$$

$$= 2\int_0^1 \frac{x^2}{2} dx$$

$$= \int_0^1 x^2 dx$$

$$= \left[\frac{x^3}{3}\right]_0^1$$

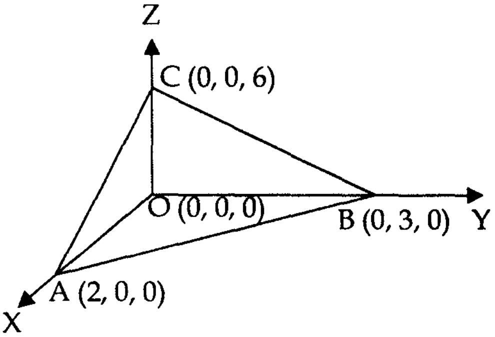
$$= \frac{1}{3}$$

**Example 7 :** Apply Stoke's theorem evaluate $\int_c \left[(x+y)dx + (2x-z)dy + (y+z)dz\right]$ where C is the boundary of the triangle with vertices (2, 0, 0), (0, 3, 0) and (0, 0, 6).

**Solution :** Hence $\vec{F} = (x+y)\hat{i} + (2x-z)\hat{j} + (y+z)\hat{k}$

$$\therefore \text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$= 2\hat{i} + \hat{k}$



Also equation of the plane through A, B, C is

$$\frac{x}{2}+\frac{y}{3}+\frac{2}{6}=1$$

or $3x + 2y + z = 6$

Let $\phi \equiv 3x + 2y + z - 6 = 0$

Normal to the plane ABC is

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial x}+\hat{k}\frac{\partial}{\partial x}\right)(3x+2y+z-6)$$

$$= 3\hat{i} + 2\hat{j} + \hat{k}$$

Unit normal vector $\hat{n} = \dfrac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{(3)^2+(2)^2+(1)^2}}$

$$= \frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}$$

$$\therefore \iint_s \operatorname{Curl}\vec{F}.\hat{n}\,ds = \iint_s \left(2\hat{i}+\hat{k}\right).\frac{1}{\sqrt{14}}\left(3\hat{i}+2\hat{j}+\hat{k}\right)ds$$

where S is the triangle ABC

$$= \frac{1}{\sqrt{14}}(6+1)\iint_s ds$$

$$= \frac{7}{\sqrt{14}}(\text{Area of } \Delta \text{ ABC})$$

It is difficult to find area of Δ ABC, so we change ds to $\dfrac{dx\,dy}{|\hat{n}.\hat{k}|}$, where $\iint dx\,dy$ is area of Δ OAB

$$=\frac{7}{\sqrt{14}}\iint_s \frac{dx\,dy}{|\hat{n}.\hat{k}|}$$

$$=\frac{7}{\sqrt{14}}\iint \frac{dx\,dy}{1/\sqrt{14}}$$

$$=7\iint_R dx\,dy$$

$$= 7\,(\text{Area of } \Delta\, OAB)$$

$$=7\left(\frac{1}{2}\times 2\times 3\right)$$

$$= 21 \qquad \text{(i)}$$

Since we know Stoke's theorem is given by

$$\oint_c \vec{F}.d\vec{r} = \iint_s \text{Curl}\,\vec{F}.\hat{n}\,ds \qquad \text{(ii)}$$

Therefore, from (i) and (ii) we have

$$\oint_c \left[(x+y)dx + (2x-z)dy + (y+z)dz\right] = 21$$

**Example 8 :** If $\vec{F} = (x-z)\hat{i} + (x^3+yz)\hat{j} - 3xy^2\hat{k}$ and S is the surface of the curve $z = a - \sqrt{x^2+y^2}$ above the xy-plane, show that $\iint_s \text{Curl}\,\vec{F}.ds = \frac{3\pi a^4}{4}$

**Solution :** Here $\vec{F} = (x-z)\hat{i} + (x^3+yz)\hat{j} - 3xy^2\hat{k}$ (i)

By Stoke's theorem, we have

$$\iint_s \text{Curl}\,F.ds = \oint_c \vec{F}.d\vec{r} \qquad \text{(ii)}$$

where S is the surface $x^2+y^2 = a^2$, z=0 above the xy-plane.

$\vec{F}.d\vec{r} = [(x-z)\hat{i} + (x^3+yz)\hat{j} - 3xy^2\hat{k}].(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$

$= (x-z)dx + (x^3+yz)dy - 3xy^2\,dz$ (iii)

Let $x = a\cos\theta$ so that $dx = -a\sin\theta\, d\theta$ and $y = a\sin\theta$ so that $dy = a\cos\theta\, d\theta$

$$\oint_c \vec{F}.d\vec{r} = \int_0^{2\pi} \left\{(a\cos\theta)(-a\sin\theta d\theta) + a^3\cos^3\theta.\, a\cos\theta d\theta\right\}$$

$\because z = 0 \therefore dz = 0$

$$= -a^2\int_0^{2\pi} \sin\theta\cos\theta\, d\theta + a^4\int_0^{2\pi}\cos^4\theta\, d\theta$$

$$= a^4\int_0^{2\pi}\cos^4\theta\, d\theta$$

$$= 4a^4\int_0^{\pi/2}\cos^4\theta\, d\theta$$

$$= 4a^4\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{5}\right)}{2\Gamma(3)}$$

$$= 4a^4\frac{\Gamma\left(\frac{1}{2}\right)\frac{3}{2}\,\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)}{4}$$

$$= a^4\frac{3}{4}\pi$$

$$= \frac{3}{4}\pi a^4 \qquad \text{(iv)}$$

$$\Rightarrow \iint_s \text{Curl}\,\vec{F}.ds = \frac{3}{4}\pi a^4$$

Using (ii) & (iv)

**EXERCISE**

1. Suppose $\vec{F} = x^3\hat{i} + y\hat{j} + z\hat{k}$ is the force field. Find the work done by $\vec{F}$ along the line from the (1, 2, 3) to (3, 5, 7).

(U.P.T.U. 2005)

Answer. 50.5 units

2. Evaluate $\int_c \vec{F}.d\vec{r}$ where $\vec{F} = xy\hat{i} + (x^2+y^2)\hat{j}$ and C is the arc of the curve $y = x^2-4$ from (2, 0) to (4,12) in the xy-plane.

Ans. 732

3. Evaluate $\int_c \vec{F}.d\vec{r}$ for $\vec{F} = 3x^2\hat{i} + (2xz-y)\hat{j} + z\hat{k}$ along the path of the carve $x^2 = 4y$, $3x^3 = 8z$ from x = 0 to x = 2

Ans. 16

4. Compute $\int_c \vec{F}.d\vec{r}$, where $\vec{F} = \dfrac{\hat{i}y - \hat{j}x}{x^2+y^2}$ and C is the circle $x^2+y^2=1$ traversed counter clockwise.

Ans. -2π

5. Find the circulation of $\vec{F}$ round the curve C, where $\vec{F} = y\,\hat{i} + z\,\hat{j} + x\,\hat{k}$ and C is the circle $x^2+y^2 = 1, z = 0$

Ans. $-\pi$

**Hint** Circulation = $\oint_c \vec{F}.d\vec{r}$

6. Find the work done in moving a particle once around a circle C in the xy-plane, if the circle has centre at the origin and radius 2 and if the force field is given by $\vec{F} = (2x-y+2z)\,\hat{i} + (x+y-z^2)\,\hat{j} + (3x-2y-5z)\,\hat{k}$

Ans. $8\pi$

7. Evaluate $\iint_s (yz\hat{i} + zx\hat{j} + xy\hat{k}).ds$ where S is the surface of the sphere $x^2+y^2+z^2 = a^2$ in the first octant.

Ans. $\frac{3a^4}{8}$

8. If $\vec{F} = (x^2-y^2)\,\hat{i} + 2xy\,\hat{j}$ and $\vec{r} = x\,\hat{i} + y\,\hat{j}$ find the value of $\int \vec{F}\cdot d\vec{r}$ around the rectangle boundary $x = 0, y = 0$ $x = a, y = b$

(U.P.T.U 2002)

Ans. $2ab^2$

9. Evaluate $\int_c \left[(\cos x \sin y - xy)dx + \sin x \cos y\, dy\right]$ by Green's theorem where C is the circle $x^2+y^2 = 1$

Ans. 0

10. Evaluate $\int_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = 4xy\,\hat{i} + yz\hat{j} - xz\hat{k}$ where and S is the surface of the cube bounded by the planes $x = 0, x = 2, y = 0, y = 2, z = 0, z = 2$

Ans. 32

11. Prove that $\iiint_v \text{Curl}\,\vec{F}\,dv = \iint \hat{n} \times \vec{F}\,ds$

12. Prove that $\int_s \hat{n} \times (\vec{a} \times \vec{r})ds = 2\vec{a}V$, where V is the volume enclosed by the surface S and $\vec{a}$ is a constant vector.

13. Verify Gauss's divergence theorem for $\vec{F} = y\,\hat{i} + x\,\hat{j} + z^2\,\hat{k}$ and S is the surface of the cylinder bounded by $x^2+y^2 = a^2$; $z = 0$; $z = h$

14. Verify divergence theorem for $\vec{F} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$ and S the surface of the cube bounded by the planes $x=0, x = 2, y = 0, y = 2, z = 0, z = 2$

15. Verify Gauss's theorem and show that

$$\int_s \left[\left(x^3 - yz\right)\hat{i} - 2x^2y\hat{j} + 2\hat{k}\right].\hat{n}\,ds = \frac{a^5}{3}$$ where S denotes the surface of the cube bounded by the planes x = 0, x = a, y = 0, y = a, z = 0, z = a.

16. Evaluate $\iint_s \left(yz\hat{i} + zx\hat{j} + xy\hat{k}\right)ds$ where S is the surface of the sphere $x^2+y^2+z^2 = a^2$ in the first octant.

(U.P.T.U 2004)

Ans. 0

17. Verify the divergence theorem for the function $\vec{F} = 2x^2y\,\hat{i} - y^2\,\hat{j} + 4xz^2\,\hat{k}$ taken over the region in the first octant bounded by $y^2+z^2 = 9$ and x = 2.

18. Evaluate by stoke's theorem $\oint\left(e^x\,dx + 2y\,dy - dz\right)$ where C is the carve $x^2+y^2 = 4$, z = 2

Ans. 0

19. Apply stoke's theorem to find the value of $\int\left(y\,dx + zdy + xdz\right)$ where C is the curve of intersection $x^2+y^2+z^2 = a^2$ and x+z = a

( JNTU 1999)

Ans. $\frac{-\pi a^2}{\sqrt{2}}$

20 Evaluate $\oint_c 2y^3dx + x^3dy + zdz$ where C is the trace of the cone intersected by the plane z = 4 and S is the surface of the cone $z = \sqrt{x^2 + y^2}$ below z = 4

(PTU 2006)

Ans. $192\pi$

21. Use Stoke's theorem to evaluate $\int \text{Curl}\,\vec{F}.\hat{n}\,ds$ over the open hemispherical surface $x^2+y^2+z^2 = a^2$, z > 0 where $\vec{F} = y\,\hat{i} + zx\,\hat{j} + y\,\hat{k}$

Hint : The boundary of hemispherical surface is circle of radius a in the plane z = 0. The parametric equations of the circle are x = a cosθ, y = b sinθ, z = 0

Here $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$, $d\,\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$, $x^2+y^2 = a^2$, z = 0

using stoke's theorem

$$\int_c \vec{F}.d\vec{r} = \int_c \left(y\hat{i} + zx\hat{j} + y\hat{k}\right).\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

we get

$$\Rightarrow \int_s \text{Curl}\,\vec{F}.\hat{n}\,ds = \int_c y\,dx$$

$$= \int_0^{2\pi} a\sin(-a\sin\theta)d\theta$$

$$= -\frac{a^2}{2}\int_0^{2\pi} 2\sin^2\theta d\theta$$

$$= -\frac{a^2}{2}\int_0^{2\pi} (1-\cos 2\theta)d\theta$$

$$= -\frac{a^2}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

= $-\pi a^2$ Answer

**Tick the Correct Answer of the Choices Given Below :-**

1. The value of $\int_c \vec{F}.d\vec{r}$ for $\vec{F} = 3x^2\hat{i} + (2xz-y)\hat{j} + z\hat{k}$ along the path of the curve $x^2 = 4y, 3x^3 = 8z$ from x = 0 to x = 2 is

(i) 8 (ii) 16
(iii) 4 (iv) 10

Ans. (ii)

2. If $\vec{F} = \frac{\hat{i}y - \hat{j}x}{x^2 + y^2}$ evaluate $\int_c \vec{F}.d\vec{r}$ where C is the circle $x^2+y^2 =1$ transversed counter clockwise

(i) $\pi$ (ii) $-\pi$
(iii) $2\pi$ (iv) $-2\pi$

Ans. (iv)

3. The value of $\int_c \vec{F}.d\vec{r}$ for $\vec{F} = x^2\hat{i} + xy\hat{j}$ and C is the curve $y^2 = x$ joining (0, 0) to (1, 1) is

(i) $\frac{7}{12}$ (ii) $\frac{3}{12}$
(iii) $\frac{5}{12}$ (iv) 1

Ans. (i)

4. If $\vec{F} = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$, Then the value of line integral $\int_c \vec{F}.d\vec{r}$ form (0, 0, 0) to (1, 1, 1) along the path $x = t, y = t^2, z = t^3$ is

(i) 2 (ii) 5
(iii) 7 (iv) $\frac{7}{2}$

Ans. (ii)

5. The value of the line integral $\int_c \left[\left(x^2+xy\right)dx+\left(x^2+y^2\right)dy\right]$,where C is the square formed by the lines y = ±1 and x = ±1 is

(i) 0 (ii) 1

(iii) -1 (iv) $\frac{1}{2}$

Ans. (i)

6. The circulation of $\vec{F}$ round the curve C, where $\vec{F} = y\,\hat{i} + z\,\hat{j} + x\,\hat{k}$ and C is the circle $x^2+y^2=1$, z = 0

(i) π (ii) -π

(iii) $\frac{\pi}{2}$ (iv) $-\frac{\pi}{2}$

Ans. (ii)

7. If $\vec{F} = 2y\,\hat{i} - z\,\hat{j} + x\,\hat{k}$ then the value of $\int_c \vec{F}\times d\vec{r}$ along the curve x= cos t, y = sin t, z =2cost from t = 0 t =π/2 is

(i) $\hat{i}+\hat{j}$ (ii) $\hat{i}-\hat{j}$

(iii) $\left(2-\frac{\pi}{4}\right)\hat{i}+\left(\pi-\frac{1}{2}\right)\hat{j}$ (iv) $\left(2-\frac{\pi}{4}\right)\hat{i}-\left(\pi+\frac{1}{2}\right)\hat{j}$

Ans. (iii)

8. The value of $\int_c \vec{F}.d\vec{r}$ where $\vec{F} = xy\,\hat{i} + (x^2+y^2)\,\hat{j}$ and C is the x axis is from x = 2 to x = 4 and the line x =4 from y =0 to y =12 is

(i) 768 (ii) 785

(iii) 763 (iv) 764

Ans. (i)

9. The circulation of $\vec{F}$ around C, where $\vec{F} = e^x \sin y\,\hat{i} + e^x \cos y\,\hat{j}$ and C is the rectangle whose vertices are (0,0), (1, 0), (1, $\frac{\pi}{2}$), (0, $\frac{\pi}{2}$) is

(i) 0 (ii) 1

(iii) -1 (iv) $-\frac{1}{2}$

Ans. (i)

10. The vector function $\overrightarrow{A}$ defined by $\overrightarrow{A}$ =(sin y+z cos x) $\hat{i}$ +(x cos y+sinz) $\hat{j}$ + (y cos z+sin x) $\hat{k}$ is irrotatinal. Then function ϕ is given by (Given $\overrightarrow{A} = \nabla\phi$ )

(i) x sin y - y sin z - z sin x + c

(ii) x sin y + y sin z + z sin x + c

(iii) x cos y + y cos z + z sin x + c

(iv) $x \cos y + y \cos z - z \sin x + c$
Ans. (ii)

11. Consider a vector field $\vec{F} = (x^2 - y^2 + x)\,\hat{i} - (2xy+y)\,\hat{j}$ Then the field is
(i) Solenoidal
(ii) irrotational
(iii) Solenoidal and irrotational
(iv) None of the above

(U.P.T.U. 2009)

Ans. (iii)

12. Consider a vector field $\vec{F} = (x^2 - y^2 + x)\,\hat{i} - (2xy+y)\,\hat{j}$. Then $\phi$ is given by:
(Given $\vec{F} = \nabla\phi$ )

(i) $\frac{x^3}{3} - xy^2 + \frac{x^2}{2} - \frac{y^2}{2}$ (ii) $\frac{x^3}{3} + xy^2 - \frac{x^2}{2} - \frac{y^2}{2}$
(iii) $\frac{x^3}{3} - xy^2 - \frac{x^2}{2} - \frac{y^2}{2}$ (iv) $\frac{x^3}{3} - xy^2 - \frac{x^2}{2} - \frac{y^2}{2}$
Ans. (i)

13. The value of $\iint_s \vec{F}.\hat{n}\, ds$ ,where $\vec{F} = z^2\,\hat{i} + xy\,\hat{j} - y^2\,\hat{k}$ and S is the surface bounded by the region $x^2+y^2 = 4$ $z = 0$, $z = 3$ is

(i) -26 (ii) 26
(iii) 13 (iv) -13
Ans. (ii)

14. If $\vec{F} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$ and S is the surface of the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$ Then the value of $\iint_s \vec{F}.\hat{n}\, ds$ is

(i) $\frac{1}{2}$ (ii) $\frac{3}{2}$
(iii) $\frac{5}{2}$ (iv) $\frac{7}{2}$
Ans. (ii)

15. If V is the volume enclosed by a closed surface S and $\vec{F} = x\,\hat{i} + 2y\,\hat{j} + 3z\,\hat{k}$. Then $\int_s \vec{F}.\hat{n}\, ds$ is
(i) 3V (ii) 6V
(iii) 9V (iv) 12V
Ans. (ii)

16. The value of $\int_s \vec{r}.\hat{n}\,ds$ ,where S is a closed surface is

(i) 3V (ii) 6V
(iii) 9V (iv) 0
Ans. (i)

17. The value of $\int_s \hat{n}\,ds$ ,where S is a closed surface is

(i) 3V (ii) 6V
(iii) 9V (iv) 0
Ans. (iv)

18. The value of $\int_s \hat{n}\cdot\left(\nabla\times\vec{F}\right)$, where $\vec{F}$ is a vector point function and S is a closed surface is

(i) 3V (ii) 6V
(iii) 9V (iv) 0
Ans. (iv)

19. The value of $\int_s \vec{F}.\hat{n}\,ds$ , for $\vec{F} = 4x\,\hat{i} - 2y^2\,\hat{j} + z^2\,\hat{k}$ taken over the region S bounded by $x^2+y^2 = 4$, $z = 0$ and $z = 3$ is

(i) $38\pi$ (ii) $84\pi$
(iii) $83\pi$ (iv) $48\pi$
Ans. (ii)

20. The value of $\int_s \vec{r}\times\hat{n}\,ds$ , for any closed surface S is

(i) 1 (ii) -1
(iii) 3 (iv) 0
Ans. (iv)

21. The value of $\int_s \vec{n}\times\nabla\phi\,ds$ ,for a closed surface S is

(i) $2\vec{a}$ (ii) $-2\vec{a}$
(iii) $3\vec{a}$ (iv) $\vec{0}$
Ans. (iv)

22. The value of $\int_s \hat{n}\times\left(\vec{a}\times\vec{r}\right)ds$ where V is the volume enclosed by the surface S and $\vec{a}$ is a constant vector is

(i) $2\vec{a}$ (ii) $2\vec{a}\,V$
(iii) $-2\vec{a}$ (iv) $\vec{0}$

Ans. (ii)

23. The value of $\int_c \left(e^{-x}\sin y\,dx + e^{-x}\cos y\,dy\right)$ where C is the rectangle with vertices $(0, 0), (\pi, 0), (\pi, \frac{\pi}{2}), (0, \frac{\pi}{2})$ is

(i) $2(e^{\pi}-1)$ (ii) $2(e^{-\pi}-1)$

(iii) $2(e^{\pi}+1)$ (iv) $2(e^{-\pi}+1)$

Ans. (ii)

24. The value of $\int_c \left(x^2y\,dx + x^2dy\right)$, where C is the boundary described counter clockwise of the triangle with vertices (0, 0), (1, 0), (1, 1) is

(i) 0 (ii) $\frac{1}{2}$

(iii) $\frac{5}{12}$ (iv) $-\frac{1}{2}$

Ans. (iii)

25. The value of $\int_c (x^2+xy)dx + (x^2+y^2)dy$ where C is the square formed by the lines y = ±1, x = ±1 is

(i) 0 (ii) $\frac{1}{2}$

(iii) $\frac{5}{12}$ (iv) $-\frac{1}{2}$

Ans. (i)

26. The value of $\int_c \left(e^x\,dx + 2y\,dy - dz\right)$, where C is the curve $x^2+y^2 = 4, z = 0$ is

(i) 0 (ii) $\frac{1}{2}$

(iii) $\frac{5}{12}$ (iv) $-\frac{1}{2}$

Ans. (i)

27. The value of the $\int_C \left(xydx + xy^2dy\right)$ where C is the square in the xy-plane with vertices (1,1) (-1, 1), (-1, -1), (1, -1) is

(i) $\frac{1}{3}$ (ii) $\frac{2}{3}$

(iii) $\frac{4}{5}$ (iv) $\frac{4}{3}$

Ans. (iv)

28. The value of $\int_{s} \vec{F}.d\vec{r}$ where $\vec{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$ and C is the boundary of the triangle with vertices at (0, 0, 0), (1, 0, 0), (1, 1, 0) is

(i) $\frac{1}{3}$ (ii) $\frac{2}{3}$
(iii) $\frac{4}{5}$ (iv) $\frac{4}{3}$
Ans. (i)

29. The value of $\int_{c} (yz\,dx + xz\,dy + xy\,dz)$, where C is the curve $x^2 + y^2 = 1$, $z = y^2$

(i) $\vec{0}$ (ii) 1
(iii) 2 (iv) -2
Ans. (i)

30. The value of $\oint_{c} \vec{r}.d\vec{r}$ is
(i) 0 (ii) 1
(iii) 2 (iv) 3
Ans. (i)

31. A necessary and sufficient condition that line integral $\int_{c} \vec{A}.d\vec{r} = 0$ for every closed curve C is that
(i) div $\vec{A} = 0$ (ii) curl $\vec{A} = 0$
(iii) div $\vec{A} \neq 0$ (iv) curl $\vec{A} \neq 0$
Ans. (ii)

32. The value of surface integral $\iint_{s} (yz\,dy\,dz + zx\,dz\,dx + xy\,dx\,dy)$, where S is the surface of the sphere $x^2+y^2+z^2=1$ is
(i) $\frac{4\pi}{3}$ (ii) 0
(iii) $4\pi$ (iv) $12\pi$
Ans. (ii)

33. If $\vec{F} = ax\hat{i} + by\hat{j} + cz\hat{k}$ a, b, c are constants, then $\iint \vec{F}.ds$ where S is the surface of a unit sphere, is
(i) 0 (ii) $\frac{4\pi}{3}(a+b+c)$
(iii) $\frac{4}{3}\pi(a+b+c)^2$ (iv) none of these
Ans. (ii)

**Indicate True or False for the following Statements:-**

1. In case f is single valued, and integral is take round a closed curve, the terminal points A and B coincide, and $f_B = f_A$
(True/False)
Ans. True

2. Green's theorem is a special case of Stoke's theorem
(True/False)
Ans. True

3. The Gauss divergence theorem is applicable for a region V if it is bounded by two closed surfaces $S_1$ and $S_2$ one which lies within the other.
(True/False)
Ans. True

4. The value of $\int_s \hat{n} \times (\vec{a} \times \vec{r}) ds$ is equal to zero where $\vec{a}$ is a constant vector.
(True/False)
Ans. False

5. For a closed surface S, the integral $\int_s \hat{n} \times \nabla\phi\, ds$ vanishes identically
(True/False)
Ans. True

6. If S be any closed surface, then $\int_s \text{Curl}\, \vec{F}.d\vec{r} = 0$
(True/False)
Ans. True

7. If $\vec{V}$ represents the velocity of a fluid particle and C is a closed curve, then the integral $\oint_c \vec{V}.d\vec{r}$ is called the circulation of $\vec{V}$ round the curve C.
(True/False)
Ans. True

8. If the circulation of $\vec{V}$ round every closed curve in a region D Vanishes, then $\vec{V}$ is said to be irrotational in D.
(True/False)
Ans. True

9. Any integral which is to be evaluated over a surface is called a surface integral.
(True/False)
Ans. True

10. Gauss divergence theorem is the relation between surface integral and line integral.
(True/False)
Ans. False

11. Gauss divergence theorem gives the relation between surface and volume integrals.
(True/False)
Ans. True
12. Stokes theorem gives the relation between surface and volume integral.
(True/False)
Ans. False
13. Stoke's theorem gives the relation between line and surface integrals.
(True/False)
Ans. True

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